Global sensitivity analysis of uncertain linear structures subject to static load

Marcos A. Valdebenito,* Iván V. González, and Héctor A. Jensen

Civil Engineering Department, Santa Maria University, Chile
*Corresponding author: marcos.valdebenito[at]usm.cl

Abstract

This contribution presents a strategy for performing global sensitivity analysis of uncertain linear structures. Global sensitivity is expressed in terms of the so-called Sobol' indices. These indices are calculated by means of Monte Carlo simulation involving substructuring and resampling. The application of the proposed framework is illustrated by means of a numerical example. The results obtained suggest that the proposed strategy can be highly efficient from a numerical viewpoint without compromising accuracy.

Keywords: Uncertain Linear Systems, Static Load, Global Sensitivity, Sobol' Indices, Substructuring, Resampling.

1 Introduction

Sensitivity analysis can provide valuable insight on the behavior of uncertain mechanical and structural systems, pinpointing its most influential input parameters [1]. Such type of information is crucial for, e.g. risk analysis and decision making [2, 3]. One approach for performing sensitivity analysis is applying variance-based measures, such as Sobol' indices [4]. These indices provide information on the fraction of the variance of the output response of a model that can be attributed to a particular random input variable or a group of input variables. The main advantages of Sobol' indices when compared to other approaches for sensitivity analysis are its relatively ease of implementation and ability to capture higher-order effects [5, 6].
The estimation of Sobol' indices is usually carried out via Monte Carlo simulation [7]. This is numerically demanding, as it comprises hundreds of thousands or even millions of deterministic structural analyses. Therefore, different approaches to circumvent this issue have been developed, including e.g. application of surrogate models [8], special sampling techniques [9], etc. In this context, this contribution presents an approach for calculating Sobol' indices most efficiently for a particular class of problems, i.e. uncertain linear structures subject to static load. The approach is based on two concepts: substructuring [10] and resampling [11]. Substructuring is applied for creating a database of precomputed solutions for portions of the structure. Then, these precomputed solutions are resampled, producing a large number of samples of the structural response, but at a fraction of the cost associated with a full structural analysis. This is due to the fact that the substructures have already been analyzed separately (i.e. precomputed); therefore, a structural analysis at the interface level suffices for calculating the structural response. In this way, it is possible to estimate the sought sensitivity indices with high numerical efficiency. The application and advantages of the proposed scheme are demonstrated through a numerical example.

2 Formulation of the Problem

2.1 Structural Model

Consider a structural or mechanical system which is modeled as linear elastic and subject to static loading and which is represented using an appropriate framework such as, e.g. the finite element method [12]. The numerical model comprises a total of N_D degrees-of-freedom and it is assumed that some of its input parameters are not known precisely and are characterized as random variables. For the sake of simplicity, it is further assumed that the uncertain parameters are related to structural properties while loading is deterministic. The uncertain parameters of the model include a total of n_θ random variables $\theta_i,\ i = 1, \ldots, n_\theta$ (collected in vector $\boldsymbol{\theta}$) whose probability distributions are $p_i(\theta_i),\ i = 1, \ldots, n_\theta$. The equilibrium equation associated with the numerical model is:

$$\mathbf{K}(\boldsymbol{\theta})\,\mathbf{u}(\boldsymbol{\theta}) = \mathbf{w} \tag{1}$$

where $\mathbf{K}(\boldsymbol{\theta})$ is the stiffness matrix, $\mathbf{u}(\boldsymbol{\theta})$ is the displacement vector and $\mathbf{w}$ is the loading vector. It is considered that the l-th component of the displacement vector is of particular interest and it is identified in the following as y, i.e. $y = y(\boldsymbol{\theta}) = u_l(\boldsymbol{\theta})$.

2.2 Global Sensitivity Analysis: Sobol' Indices

A possible means for characterizing the impact of the uncertain input parameters $\boldsymbol{\theta}$ on the response of interest y is carrying out a variance-based sensitivity analysis. This means that one is interested in determining the portion of the variance associated with y that can be attributed to each uncertain parameter θ_i (or groups of them). This framework for carrying out sensitivity analysis has been termed in the literature as Sobol' indices [7]. The framework consists of decomposing $y(\boldsymbol{\theta})$ into a summation of functions of increasing dimensionality with respect to $\boldsymbol{\theta}$ such that the variance of y is expressed as a summation of conditional variances.

Sobol' indices allow determining the fraction of the response variance that can be attributed either to an input variable on its own (first-order indices) or to an input variable and its interaction with other variables (total indices). In the more general case, Sobol' indices allow determining the fraction of the response variance that can be attributed to a group of random input variables. In this contribution, the focus is on first-order and total indices.

The first order Sobol' index, denoted as S_i, is equal to the variance of the expected value of the response of interest conditioned on θ_i normalized by the total variance of the response. Mathematically, S_i is defined as follows.

$$S_i = \frac{\mathbb{V}_{\theta_i}\left[\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[y|\theta_i\right]\right]}{\mathbb{V}_{\boldsymbol{\theta}}\left[y\right]},\ i = 1,\ldots,n_\theta \tag{2}$$

In the above equation, $\mathbb{E}_{\mathbf{x}}$ and $\mathbb{V}_{\mathbf{x}}$ denote expected value and variance operators calculated with respect to the set of random variables $\mathbf{x}$, respectively, while $\boldsymbol{\theta}_{-i}$ denotes the set of all random input variables associated with the model except for θ_i. The total Sobol' index, denoted as S_{T_i}, is defined as the complement of the variance of the expected value of the structural response conditioned on all random input variables of the model except θ_i, normalized by the total variance.

$$S_{T_i} = 1 - \frac{\mathbb{V}_{\boldsymbol{\theta}_{-i}}\left[\mathbb{E}_{\theta_i}\left[y|\boldsymbol{\theta}_{-i}\right]\right]}{\mathbb{V}_{\boldsymbol{\theta}}\left[y\right]},\ i = 1,\ldots,n_\theta \tag{3}$$

2.3 Calculation of Sobol' Indices

It can be noted from eqs. (2) and (3) that the evaluation of Sobol' indices demands calculating expected values and variances of the structural response conditioned on different subsets of the uncertain input parameters. In turn, this means that a number of integrals involving the structural response and subsets of the uncertain variables must be calculated. In general, the number of integrals to be calculated can be large. Furthermore, for cases of practical interest, the response of interest y cannot be calculated analytically, as seen from eq. (1). The latter two issues favor the application of the Monte Carlo method for calculating Sobol' indices [7]. The procedure to calculate the indices is the following (see, e.g. [1]).

1. Generate two independent sets $\mathbf{A}$ and $\mathbf{B}$ comprising N independent samples of $\boldsymbol{\theta}$ distributed according to $p_i(\theta_i),\ i = 1,\ldots,n_\theta$. Each of these sets is actually a matrix of dimensions $n_\theta \times N$, i.e. each column of matrices A and B contains a sample of the input parameters of the structural model. The value N should be chosen such that convergence of Monte Carlo integration is ensured [7], e.g. $N = 10^5$.

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{\theta}_{\boldsymbol{A}}^{(1)} & \boldsymbol{\theta}_{\boldsymbol{A}}^{(2)} & \cdots & \boldsymbol{\theta}_{\boldsymbol{A}}^{(j)} & \cdots & \boldsymbol{\theta}_{\boldsymbol{A}}^{(N)} \end{bmatrix} \tag{4}$$

$$\mathbf{B} = \begin{bmatrix} \boldsymbol{\theta}_{\boldsymbol{B}}^{(1)} & \boldsymbol{\theta}_{\boldsymbol{B}}^{(2)} & \cdots & \boldsymbol{\theta}_{\boldsymbol{B}}^{(j)} & \cdots & \boldsymbol{\theta}_{\boldsymbol{B}}^{(N)} \end{bmatrix} \tag{5}$$

2. Define matrix $\boldsymbol{C}_i,\ i = 1,...,n_\theta$ equal to $\boldsymbol{B}$ except for the i-th row, which is equal to the i-th row of matrix $\boldsymbol{A}$. That is:

$$\boldsymbol{C}_i = \begin{bmatrix} \boldsymbol{\theta}_{\boldsymbol{C}_i}^{(1)} & \boldsymbol{\theta}_{\boldsymbol{C}_i}^{(2)} & \cdots & \boldsymbol{\theta}_{\boldsymbol{C}_i}^{(N)} \end{bmatrix}, \qquad i = 1,\ldots,n_\theta \tag{6}$$

$$\boldsymbol{\theta}_{\boldsymbol{C}_i}^{(j)} = \begin{bmatrix} \theta_{1\mathbf{B}}^{(j)} & \theta_{2\mathbf{B}}^{(j)} & \cdots & \theta_{i\mathbf{A}}^{(j)} & \cdots & \theta_{n_\theta\mathbf{B}}^{(j)} \end{bmatrix}^T, \qquad j = 1,\ldots,N \tag{7}$$

3. Calculate the response of interest y for each set of samples (i.e. each column) of matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}_i,\ i=1,\ldots,n_\theta$.

$$y_{\boldsymbol{A}}^{(j)} = f\left(\boldsymbol{\theta}_{\boldsymbol{A}}^{(j)}\right) \quad y_{\boldsymbol{B}}^{(j)} = f\left(\boldsymbol{\theta}_{\boldsymbol{B}}^{(j)}\right) \quad y_{\boldsymbol{C}_i}^{(j)} = f\left(\boldsymbol{\theta}_{\boldsymbol{C}_i}^{(j)}\right),\ i=1\ldots,n_\theta,\ j=1,\ldots,N \tag{8}$$

4. Calculate the following estimators.

$$f_0 = \frac{1}{N}\sum_{j=1}^{N} y_{\boldsymbol{A}}^{(j)} \qquad D = \frac{1}{N}\sum_{j=1}^{N} {y_{\boldsymbol{A}}^{(j)}}^2 - {f_0}^2 \tag{9}$$

$$U_{S_i} = \frac{1}{N}\sum_{j=1}^{N} y_{\boldsymbol{A}}^{(j)} y_{\boldsymbol{C}_i}^{(j)} \qquad U_{T_i} = \frac{1}{N}\sum_{j=1}^{N} y_{\boldsymbol{B}}^{(j)} y_{\boldsymbol{C}_i}^{(j)} \tag{10}$$

5. The sought Sobol' indices are estimated applying the following formulas.

$$S_i = \frac{U_{S_i} - {f_0}^2}{D} \qquad T_i = 1 - \frac{U_{T_i} - {f_0}^2}{D},\ i=1,\ldots,n_\theta \tag{11}$$

As noted from the procedure described above, the structural response must be evaluated a total of $N_T = N\left(n_\theta + 2\right)$ times considering all samples of the input structural parameters contained in sets $\mathbf{A}$, $\mathbf{B}$ and $\boldsymbol{C}_i,\ i=1,\ldots,n_\theta$. Hence, the total number of samples of the structural response N_T is usually quite large in practical applications, e.g. $N_T > 10^6$. Undoubtedly, this imposes a huge numerical burden, as a prohibitive number of structural analyses must be carried out. Therefore, in the following, an approach for alleviating this numerical burden is discussed.

3 Proposed Approach

3.1 General Remarks

This section discusses an approach that allows to generate a large number of samples of the structural response most efficiently; then, these samples can be considered to calculate Sobol' indices. The proposed approach combines substructuring and resampling. Each of these steps is discussed separately below.

3.2 Static Substructuring

Assume that each component of vector $\boldsymbol{\theta}$ affects a distinct portion of the structural system. Then, it is possible to separate the complete structure into n_θ substructures connected between them by interfaces. Each substructure contains internal degrees-of-freedom, denoted by index I, and boundary (or interface) degrees-of-freedom, denoted by index B. Hence, the equilibrium equation associated with substructure i can be written using block matrix notation as follows [10].

$$\begin{bmatrix} \mathbf{K}_{B,B}^{(i)}\left(\theta_i\right) & \mathbf{K}_{B,I}^{(i)}\left(\theta_i\right) \\ \mathbf{K}_{I,B}^{(i)}\left(\theta_i\right) & \mathbf{K}_{I,I}^{(i)}\left(\theta_i\right) \end{bmatrix} \begin{Bmatrix} \mathbf{u}_B^{(i)}\left(\boldsymbol{\theta}\right) \\ \mathbf{u}_I^{(i)}\left(\boldsymbol{\theta}\right) \end{Bmatrix} = \begin{Bmatrix} \mathbf{w}_B^{(i)} \\ \mathbf{w}_I^{(i)} \end{Bmatrix} \tag{12}$$

The preceding equation can be expressed in terms of the boundary degrees-of-freedom applying static condensation [10, 13], yielding:

$$\hat{\mathbf{K}}^{(i)}\left(\theta_i\right)\mathbf{u}_B^{(i)}\left(\boldsymbol{\theta}\right) = \hat{\mathbf{w}}^{(i)}\left(\theta_i\right) \tag{13}$$

where $\hat{\mathbf{K}}^{(i)}\left(\theta_i\right)$ and $\hat{\mathbf{w}}^{(i)}\left(\theta_i\right)$ are the condensed stiffness matrix and condensed load vector associated with substructure i and are defined as:

$$\hat{\mathbf{K}}^{(i)}\left(\theta_i\right) = \mathbf{K}_{B,B}^{(i)}\left(\theta_i\right) - \mathbf{K}_{B,I}^{(i)}\left(\theta_i\right)\left(\mathbf{K}_{I,I}^{(i)}\left(\theta_i\right)\right)^{-1}\mathbf{K}_{I,B}^{(i)}\left(\theta_i\right) \tag{14}$$

$$\hat{\mathbf{w}}^{(i)}\left(\theta_i\right) = \mathbf{w}_B^{(i)} - \mathbf{K}_{B,I}^{(i)}\left(\theta_i\right)\left(\mathbf{K}_{I,I}^{(i)}\left(\theta_i\right)\right)^{-1}\mathbf{w}_I^{(i)} \tag{15}$$

The condensed stiffness matrix and load vector of all substructures can be assembled in order to produce an equilibrium equation than involves all boundary degrees-of-freedom of the structure [10, 13]:

$$\hat{\mathbf{K}}\left(\boldsymbol{\theta}\right)\mathbf{u}_B\left(\boldsymbol{\theta}\right) = \hat{\mathbf{w}}\left(\boldsymbol{\theta}\right) \tag{16}$$

where $\hat{\mathbf{K}}$ denotes the stiffness matrix associated with the boundary degrees-of-freedom while $\hat{\mathbf{w}}$ and $\mathbf{u}_B$ are the load and displacement vectors associated with the boundary degrees-of-freedom, respectively. Once the displacements $\mathbf{u}_B$ have been determined by means of eq. (16), it is possible to determine the displacements at the internal degrees-of-freedom of the i-th substructure by means of the following expression:

$$\mathbf{u}_I^{(i)}(\boldsymbol{\theta}) = \left(\mathbf{K}_{I,I}^{(i)}(\theta_i)\right)^{-1}\left(\mathbf{w}_I^{(i)} - \mathbf{K}_{I,B}^{(i)}(\theta_i)\,\mathbf{u}_B^{(i)}(\boldsymbol{\theta})\right) \tag{17}$$

It should be noted that calculating the displacement vector applying either eq. (1) or eqs. (16) and (17) is completely equivalent, as it produces identical results and demands the same amount of numerical efforts. However, the application of substructuring may become advantageous in context with resampling, as described in the next section.

3.3 Resampling

A possible means for reducing the numerical costs associated with the calculation of N samples of the structural response is the application of resampling in combination with substructuring [11]. The main idea behind this procedure is the following. In a first stage, a relatively small number N_A of samples of the uncertain input parameters $\boldsymbol{\theta}^{(j)}$, $j = 1, \ldots, N_A$ is generated, such that $N_A < N$. For each of these samples, structural analysis is carried out at the substructure level. That is, the condensed stiffness and load vectors are generated in terms of the boundary degrees-of-freedom (see eqs. (14) and (15), respectively). Then, in a second stage, these stiffness matrices and load vectors associated with each substructure are assembled in order to calculate the displacements at both the boundary and internal degrees-of-freedom (see eqs. (16) and (17), respectively). Nonetheless, it is at this second stage where the concept of resampling comes into play: at the moment of assembling the stiffness matrix and load vector, the data required from the substructure level is chosen at random (with replacement) from the available set of N_A samples generated at the first stage. The advantage of applying resampling is that it is possible to generate a total number of samples of the structural response N which is larger than the total number of samples of each substructure N_A as a single sample of a substructure is used more than once. In fact, the application of this scheme would allow – in theory – to generate a total of $N_A^{n_\theta}$ samples of the structural response. The whole idea of the resampling scheme is illustrated schematically in fig. 1, where it is assumed that a structure is modeled considering two substructures ($n_\theta = 2$), while $N_A = 2$ and $N = 3$.

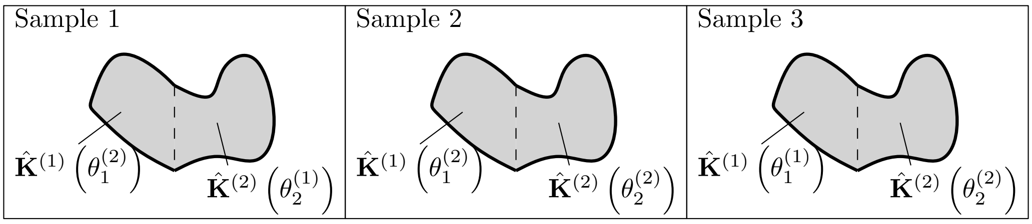

Figure 1: Schematic representation of substructuring and resampling

The main advantage of the scheme for generating samples of the structural response based on substructuring and resampling is the following. The calculation of N of the response demands performing structural analysis at the substructure level a total of N_A times plus additional N structural analysis at the boundary degrees-of-freedom. Recalling that $N_A \ll N$ and that structural analysis at the level of the boundary degrees-of-freedom (see eq. (16)) can be numerically less demanding than performing a full system analysis (see eq. (1)), the proposed scheme can report important savings in numerical efforts.
It should be noted that the resampling scheme described above produces samples of the structural response which possess a degree of correlation. This is due to the fact that one sample of a substructure is considered more than once. While this correlation does not introduce bias in the estimator of the Sobol' indices, it may increase its variability [14]. At the moment, the quantification of the effect of this correlation remains an open challenge.

4 Example

The application of the proposed scheme for calculating Sobol' indices that combines substructuring and resampling is illustrated by means of the following example. The example considered herein is a simplified model of a human tooth, as depicted in fig. 2, which has been adapted from [15]. The finite element model of the tooth is developed under the assumption of a plane strain state and it involves a total of 15312 degrees-of-freedom. The anatomy of a human tooth can be modeled considering five different layers: bone, peridontal ligament, enamel, dentine and pulp. Due to several reasons (age, nutrition, dental hygiene, etc.), there is uncertainty concerning the elastic properties that characterize the behavior of the different layers. Hence, the Young's moduli of each layer are characterized considering truncated Gaussian distributions with mean value and coefficient of variation as shown in Table 1; please note that the statistics reported in the Table correspond to the Gaussian distributions before truncation. The characterization of the uncertainty associated with the Young's moduli is based on the data reported in [15–20]. The loading over the model consists of a deterministic horizontal point load of 1 [N] that simulates the effect of an orthodontics appliance. The response of interest is the horizontal displacement of the cusp of the tooth and hence, both first-order and total Sobol' indices associated with this response are calculated.

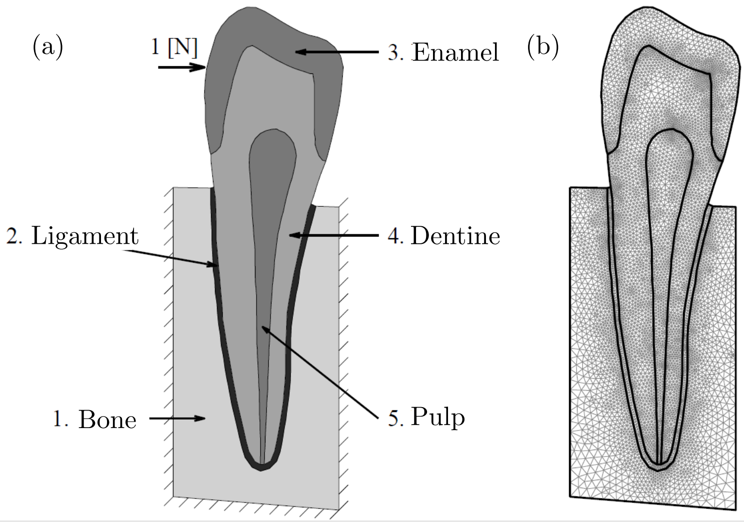

Figure 2: (a) Human tooth subject to static loading (b) finite element model

Table 1: Parameters of the human tooth model

Tooth layer	Poisson's ratio	Young's modulus	
		Mean value μ_E [MPa]	Coefficient of Variation
1 Bone	0.30	14000	8%
2 Ligament	0.45	100	20%
3 Enamel	0.30	80000	10%
4 Dentine	0.31	17000	18%
5 Pulp	0.49	4	30%

The Sobol' indices are estimated first applying Monte Carlo simulation considering a total of $N = 10^5$ samples. No substructuring or resampling schemes are applied and thus, the structural response is calculated by repeatedly solving eq. (1) for different samples of the input parameters. The results obtained are depicted in fig. 3 and are shown with dashed line. Please note that in this figure, the curves associated with the sensitivity indices of bone, enamel and pulp overlap, as their overall effect on the response under study is negligible.

In a second step, Sobol' indices are estimated applying the substructuring and resampling scheme. Five substructures are considered for the analysis and each substructure matches the different layers of the tooth model. The number of samples of each substructure is set equal to $N_A = 10^3$; then, a total of $N = 10^5$ samples are generated by means of resampling. The results obtained are depicted in fig. 3 with solid line. This figure suggests an excellent agreement between the reference results and the results obtained with substructuring and resampling, as both lead to almost identical estimates of the first-order and total Sobol' indices. Furthermore, the proposed approach for generating samples of the structural responses reduces the numerical costs considerably. In fact, a speedup factor of 2500 was observed when comparing the proposed approach with that where no substructuring and resampling are considered.

The results obtained reveal that the variability of the horizontal displacement of the tooth's cusp is highly

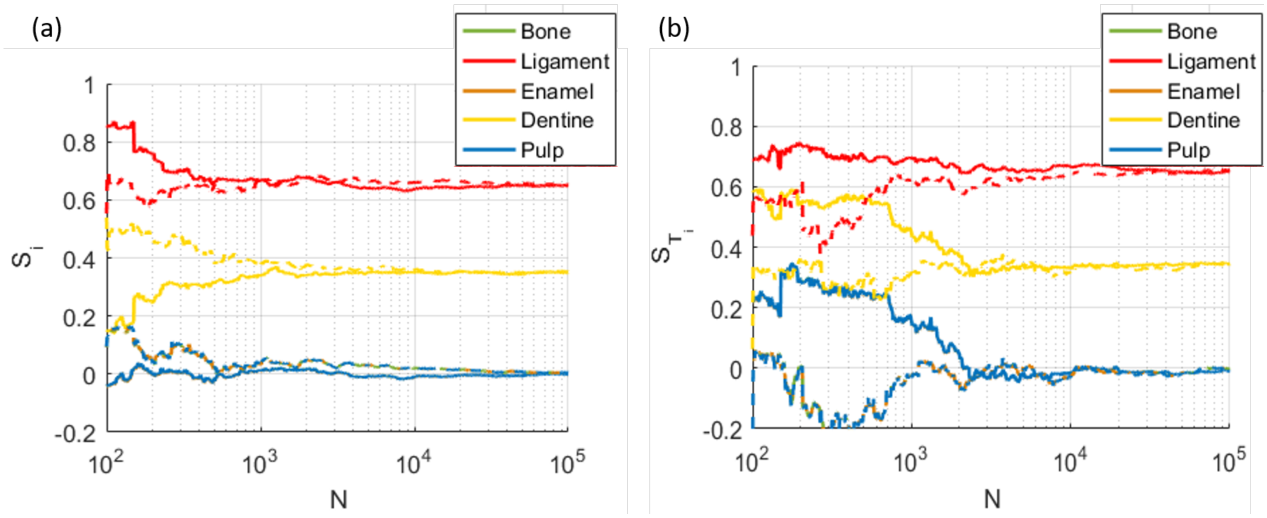

Figure 3: (a) First-order and (b) total Sobol' indices estimates. Solid line: proposed approach combining substructuring and resampling; dashed line: direct simulation (considering neither substructuring nor resampling).

influenced by the ligament. This make sense from a physical viewpoint, as both the bone and dentine are relatively rigid layers while the ligament that connects them is quite flexible.

5 Conclusions

The results presented in this contribution suggest that the application of substructuring and resampling for estimating Sobol' indices can lead to substantial savings in numerical efforts without compromising accuracy. However, there are several practical issues that should be further investigated. For example, at the moment there is no criterion available for the selection of the number of substructures N_A to be resampled. In addition, the application of resampling induces a correlation in the samples of the structural response that should be quantified. Furthermore, the possibility of accounting for more than one uncertain parameter per substructure should be addressed as well. All of these issues are currently under investigation.

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