Influence of uncertainty and numerical errors in the context of MIMO systems

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Abstract

Multiple-input multiple-output (MIMO) systems can be considerably afflicted by uncertainty and numerical errors from various sources at different stages in the modeling and simulation process. In this paper, we perform a verification and validation analysis of the process for frequency selective MIMO systems and establish the respective current verification degree for the subtasks. Our special focus is on the stage of channel estimation, for which we show in detail how the verification degree can be improved using methods with result verification and analyse the uncertainty using Monte-Carlo and interval techniques. However, we also touch upon the stages of channel simulation and power allocation from the same point of view.

Keywords: Verification and validation, Methods with result verification, Simulation, frequency selective MIMO, Channel estimation

1 Introduction

To demonstrate how development in modern engineering works, it is commonly accepted nowadays [1] to consider the so called modeling and simulation cycle [2]. It comprises the three phases analysis, formalization and simulation, which can be reiterated in this order a number of times, the process known as the optimization (calibration) phase. The same cycle can be used to illustrate the verification and validation (V&V) activities which should accompany the development process. Devising and testing methodologies and tools for V&V assessment is a long-standing tradition originally from the area of computational fluid dynamics, supported by the American Society of Mechanical Engineers. The terms verification and validation are defined in the context of modeling and simulation tasks, of software engineering, and of numerical mathematics. Basically, the verification activities are those performed on the way from the formalization to the simulation to make sure that the implementation and its result are correct. Validation is performed en route from the simulation to the new analysis iteration to determine whether the results of the implementation have a good correspondence to reality, whatever 'good' and 'reality' mean in a given engineering context.

The researches in the V&V area develop compilations of requirements for users to categorize and classify processes simply by employing an assessment procedure or a criterion from a catalog [1, 3, 4]. The known methodologies usually rely on analytically solvable special cases, simplifications and benchmark examples if finding theoretical proofs is too difficult. That is, they do not provide a definitive step-by-step V&V procedure immediately applicable by the engineer. A possible step in this direction was described in [5, 6]. The major difference to the usual V&V methodologies is at the level of verification. Without developing specialized verification benchmarks to test the process, we suggest to perform the verification stage for the appropriately decomposed process itself (or at least, its vital parts), for example, using methods with result verification [7]. Such methods provide a mathematical guarantee that the results obtained on a computer – which, in the usual case, can turn out to be completely inaccurate due to finite precision of the computer arithmetic – are inside a certain region and not outside it. There are several successful examples of such computerized proofs, for example, [8]. As a kind of a by-product, we can propagate bounded uncertainty in parameters through systems using methods with result verification. Moreover, the areas of qualification/analysis and validation/simulation can profit from their use. For example, it can be demonstrated with their help whether the (numerical) formal model is chosen adequately, whether the parameter ranges are determined correctly, or whether the parameters to be neglected are selected appropriately.

In the times when wireless communications are omnipresent, multiple-input multiple-output (MIMO) systems play an important role. MIMO is well established in many wireless and wired communication standards and is used to increase the channel capacity without the need to increase the channel bandwidth or the transmit power [9]. The channel capacity is the information theoretic limit of the bit rate, that is, the number of bits per second, that can be transmitted through a physical channel error free. With MIMO, multiple data streams are transmitted on the same frequency band and at the same time. For separating the data streams, spatially separated channels are used, which can be achieved, for example, by placing multiple antennas at the transmitter and receiver side at different locations. Signal processing techniques such as singular value decomposition (SVD) are then used to remove the occurring inter-channel as well as inter-symbol interferences.

MIMO systems can be considerably afflicted by uncertainty and numerical errors from various sources at different stages in the modeling and simulation process. There are several conventional studies dealing with this problem especially at the stage of channel estimation, see for example, [9–12]. Moreover, there are several contributions dealing with formal verification of systems from the area of signal processing. For example, in the report [13], the authors give details about the ongoing work on formalization and property proofs of FAUST (Functional Audio Stream) programs using the tool COQ (`coq.inria.fr`). Note that formal verification (formal proofs, model checking) is different from result verification mentioned above since its goal is to prove the correctness of algorithms with respect to a certain formal property, usually using logic (or so called formal methods of mathematics). In this sense, formal verification and result verification complement each other and do not compete at the stage of verification. A reference of how to use interval methods [7] for identification of MIMO autoregressive systems with exogenous input is in [14].

All this work notwithstanding, a systematical treatment of MIMO modeling and simulation from the point of view of V&V still seems to be lacking. In this paper, we analyse the task of digital wireless communications using V&V techniques as described in [6]. A short overview of main concepts from the area of V&V is in Section 2. Our special focus in this paper is on the verification stage. In particular, we analyse in detail the influence of uncertainty and numerical errors for the subtask of broadband MIMO channel estimation, which we address using methods with result verification in Section 4. Moreover, we give a general overview of what needs to be done for verifying the subtasks at the power allocation and channel simulation stages in Section 3. A summary of the main results and our future work is in Section 5.

2 V&V: A Short Overview

Standard terminology [1] defines verification as the process of determining that a model implementation accurately represents the developers conceptual description of the model and the solution to the model. Validation is understood as the process of determining the degree to which a model is an accurate representation of the real world from the perspective of its intended uses. Note that this terminology, although widely spread, is not always used correctly or consistently. For example, the term 'channel verification' is actually used to mean 'channel validation' in [11]; the title 'Validated Numerics' in [15] should be 'verified numerics' according to the definitions above.

In [5], a classification of computer models (or simulations, abridged as CM in the following) associated with a given system (process) was proposed for easy identification of the verification degree.

- C4: CM uses standard floating point or fixed point arithmetic (the lowest verification degree).
- C3: CM is subdivided into appropriate subtasks and uses at least standardized IEEE 754 floating point arithmetic. Additionally, sensitivity analysis is carried out or uncertainty is quantified using traditional methods (e.g., Monte Carlo).
- C2: Relevant subtasks are implemented using tools with result verification or delivering reliable error bounds. The convergence of numerical algorithms is proved via analytical solutions, computer-aided proofs, or fixed point theorems.
- C1: The whole CM is verified using tools and algorithms with result verification or using real number algorithms, analytical solutions, or computer-aided existence proofs. Uncertainty is quantified and propagated throughout the CM using verified or stochastic (or both) approaches; sensitivity analysis is carried out.

The degree is tagged with a minus if the sensitivity analysis and uncertainty quantification were not carried out. A plus alongside a degree means that the code verification (e.g, using literate programming [16]) is performed additionally. For example, a CM categorized as having degree C3- corresponds to a situation common in small-scale engineering applications, where the model and the simulation use IEEE 754 floating point software and neither the sensitivity analysis nor uncertainty quantification need to be carried out.

Degree C3+ means that, in addition to C3, the code of the CM was verified, that is, it was proven somehow that the program for the model did not contain any implementation errors. Note that it does not necessarily mean that the result of a simulation with such a CM would be correct.

In [17, 18], a questionnaire was developed for determining the accuracy and correctness issues of a process and tested in a biomechanical context. This questionnaire is also useful for MIMO systems (cf. Section 4.1) since it helps to regulate the V&V assessment and to assign/improve the verification degree of the CM. In general, a typical questionnaire should aid in organizing a complex process with respect to initial data sources, their accuracy and its parameter and result ranges. Moreover, it should require precise description of special functions, algorithms (e.g., as a UML activity diagram), and employed mathematical operations. Finally, it should help to find an optimal output data format (or the appropriate means of data exchange).

3 Digital MIMO Communications from the Angle of V&V

In wireless systems, placing multiple antennas at the transmitter and receiver sides improves both the capacity and the integrity of a communications link, a strategy known as the MIMO method [19, 20]. A model for a frequency selective MIMO link consisting of n_T transmitting and n_R receiving antennas is

$$\mathbf{y}_k = \sum_{i=0}^{L_\mathrm{h}-1} H_i \cdot \mathbf{c}_{k-i} + \mathbf{n}_k, \qquad \mathbf{y}_k, \mathbf{n}_k \in \mathbb{C}^{n_\mathrm{R}},\ \mathbf{c}_{k-i} \in \mathbb{C}^{n_\mathrm{T}},\ H_i \in \mathbb{C}^{n_\mathrm{R}\times n_\mathrm{T}} \ , \tag{1}$$

where $k \in \mathbb{Z}$ is a specific instant of time, L_h is the number of channel taps per I/O path, $\mathbf{y}_k$ is the received data vector, $\mathbf{c}_{k-i}$ is the transmitted signal vector, $\mathbf{n}_k$ is the additive white Gaussian noise vector at the receiver side with a variance σ in both real and imaginary parts, and H_i is the channel matrix.

To work with this model, solving the subtasks shown in Figure 1 is necessary. After the system configuration is established, the channel matrices H_i need to be estimated. After that, the frequency selective MIMO link is decomposed into a number of independent weighted frequency flat single-input single-output (SISO) links, for example, using the SVD technique [21]. The weights of such SISO links are usually not equal so that a need for external power allocation might arise. The goal during the power allocation stage is to optimize a certain quality criterion, for example, to minimize the bit error rate (BER). One possibility for that is to use the Lagrange multipliers method. In the following subsections, we describe the models for each of the subtasks in more detail.

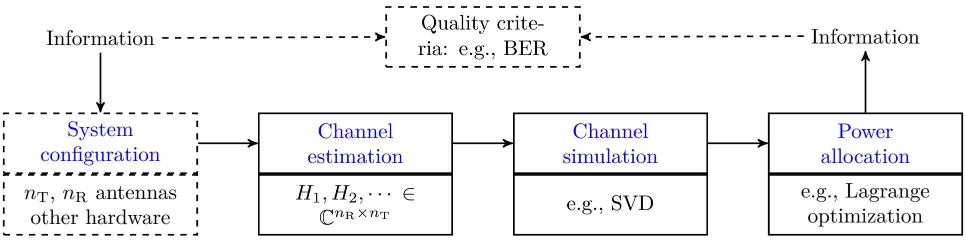

Figure 1: Subprocesses during the modeling and simulation of a (MIMO) digital channel

3.1 Channel Estimation Stage

The first stage in the process is the channel estimation stage. One possibility for identifying the values of the channel matrices H_i is based on the least squares minimization. For frequency selective MIMO channels, a so called pilot sequence of a certain length L_s is chosen for each MIMO input first. Usually, a sequence fulfilling a certain optimality condition should be preferred, but a random sequence of (complex) values can also be chosen. Denote by S the matrix containing pilot sequence vectors which are transmitted at the inputs $\mu = 1 \ldots n_\mathrm{T}$. Further, denote by $\mathbf{r}_\nu$ the corresponding received signal for each output $\nu = 1 \ldots n_\mathrm{R}$. The following model describes the interconnection between the received and transmitted signals:

$$\mathbf{r}_\nu = (S_1\ S_2\ \ldots\ S_{n_\mathrm{T}}) \cdot \begin{pmatrix} \mathbf{h}_{\nu 1} \\ \vdots \\ \mathbf{h}_{\nu n_\mathrm{T}} \end{pmatrix} + \mathbf{n}_\nu \text{ or } \mathbf{r}_\nu = S\mathbf{h}_\nu + \mathbf{n}_\nu \ , \tag{2}$$

where $\mathbf{n}_\nu$ is the white Gaussian noise vector and $\mathbf{h}_{\nu\mu} \in \mathbb{C}^{L_\mathrm{h}}$. Here, $\mathbf{h}_{\nu\mu}$ denotes the L_h symbol-spaced channel taps between the MIMO input μ and the output ν. This system has $(L_\mathrm{s} - L_\mathrm{h} + 1)$ equations and $n_{\mathrm{T}}L_{\mathrm{h}}$ unknowns, where L_{h} can also be understood as the number of matrices H_i to be estimated. That is, to be able to get a practically meaningful solution, the condition $L_{\mathrm{s}} - L_{\mathrm{h}} + 1 \geq n_{\mathrm{T}}L_{\mathrm{h}}$ should be fulfilled. That means that the minimum length of the pilot sequence is $L_{\mathrm{s}} = n_{\mathrm{T}}L_{\mathrm{h}} + L_{\mathrm{h}} - 1$. In the linear system (2), the response $\mathbf{r}_\nu$ and the information S can be considered as known. Moreover, $\mathbf{n}_\nu$ has a known density with the zero mean and variance σ. The channel matrices H_i consisting of $\mathbf{h}_\nu$ need to be estimated, the exact relationship between $\mathbf{h}_{\nu\mu}$ and H_i being

$$H_i = \begin{pmatrix} \mathbf{h}_{11}[i] & \cdots & \mathbf{h}_{1n_{\mathrm{T}}}[i] \\ \vdots & \ddots & \vdots \\ \mathbf{h}_{n_{\mathrm{R}}1}[i] & \cdots & \mathbf{h}_{n_{\mathrm{R}}n_{\mathrm{T}}}[i] \end{pmatrix} ,$$

where the notation $\mathbf{h}_{\nu\mu}[i]$ means the i-th component of the vector $\mathbf{h}_{\nu\mu}$. Since the system can be overdetermined, the optimal solution in the sense of the least squares minimization can be found by the following procedure (S^{H} is the conjugate transposed of S):

$$\begin{aligned} \mathbf{r}_\nu + \mathbf{n}_\nu = S\mathbf{h}_\nu &\Leftrightarrow S^{\mathrm{H}}(\mathbf{r}_\nu + \mathbf{n}_\nu) = S^{\mathrm{H}}S\mathbf{h}_\nu \\ &\Leftrightarrow (S^{\mathrm{H}}S)^{-1}S^{\mathrm{H}}(\mathbf{r}_\nu + \mathbf{n}_\nu) = (S^{\mathrm{H}}S)^{-1}S^{\mathrm{H}}S\mathbf{h}_\nu \\ &\Leftrightarrow \mathbf{h}_\nu = (S^{\mathrm{H}}S)^{-1}S^{\mathrm{H}}(\mathbf{r}_\nu + \mathbf{n}_\nu). \end{aligned} \quad (3)$$

Numerically more feasible is solving the system of linear matrix equations $S^{\mathrm{H}}(R + N) = S^{\mathrm{H}}SH$, where $R = (\mathbf{r}_\nu)$ for all ν and N is the matrix of noise. In the presence of the Gaussian noise, it is advantageous to choose pilot sequences with higher lengths for achieving more accurate approximations for H. On the other hand, higher pilot lengths reduce the capacity for transmitting real, useful information [12].

Pilot sequences can be chosen such that $S^{\mathrm{H}} \cdot S = \alpha I$. Here, I is the identity matrix, $\alpha = (L_{\mathrm{s}} - L_{\mathrm{h}} + 1) \cdot U_{\mathrm{s}}^2$ and U_{s} is the half level transmit amplitude. This optimality condition simplifies formula (3) even further:

$$\mathbf{h}_\nu = \frac{1}{(L_{\mathrm{s}} - L_{\mathrm{h}} + 1)U_{\mathrm{s}}^2} \cdot S^{\mathrm{H}} \left(\mathbf{r}_\nu + \mathbf{n}_\nu\right) \ . \quad (4)$$

More complex models can be considered, but for the purposes of the first study, we analyse this simple estimation procedure from the point of view of V&V. Since this analysis is in the focus of this paper, we give the details separately in Section 4. There, we show how to improve the overall verification degree to C2- for this part of the process.

3.2 Channel Simulation Stage

For ease of presentation, we consider w.l.o.g. $L_{\mathrm{h}} = 1$ in this section and in Section 3.3. The resulting frequency flat model can be readily extended to frequency selective channel conditions as shown in [21]. To obtain a number of independent, weighted SISO links from a given MIMO channel $H = H_0$, a well-established SVD technique can be used. It is assumed that the matrix H can be decomposed as $H = U \cdot \Sigma \cdot V^{\mathrm{H}}$. Here, U and V are unitary matrices and Σ is the diagonal matrix with real elements. The matrix Σ contains the positive square roots of the eigenvalues of $H^{\mathrm{H}}H$ in descending order on the main diagonal. They are called singular values and denoted by $\sqrt{\xi_i}$ in the following. If we consider a pre-processed data vector $\mathbf{x} := V \cdot \mathbf{c}$ and post-process the corresponding receive signal $\mathbf{z} := H\mathbf{x} + \mathbf{n}$ by multiplying it by U^{H}, then

$$\mathbf{u} := U^{\mathrm{H}}\mathbf{z} = U^{\mathrm{H}} \underbrace{(U\Sigma V^{\mathrm{H}})}_{H} V\mathbf{c} + \underbrace{U^{\mathrm{H}}\mathbf{n}}_{:=\mathbf{w}} = \Sigma\mathbf{c} + \mathbf{w} \ . \quad (5)$$

In this way, the MIMO system is (theoretically) transformed into $L = \min\{n_{\mathrm{T}}, n_{\mathrm{R}}\}$ independent, non-interfering layers u_i having unequal weights $\sqrt{\xi_i}$:

$$u_i = \sqrt{\xi_i}c_i + w_i \quad \text{for} \quad i = 1 \ldots L \ . \quad (6)$$

Other methods for achieving the same goal of separating the SISO links can be employed, for example, the geometric mean decomposition (GMD).

There are a number of difficulties during this stage due to numerics or uncertainty. For example, $U^{\mathrm{H}} \cdot U$ or $V^{\mathrm{H}} \cdot V$ are not exactly identity matrices due to numerical errors, creating interference between the links. Moreover, the matrix H is strictly speaking uncertain, making the weights also uncertain. It is therefore a challenging and important task to quantify their influence at this stage. The verification degree is C3- at the moment.

The improvement of the verification degree is challenging since it is not easy to perform this stage using methods with result verification, especially if SVD is employed. There is a certain amount of research on bounding real singular values of (interval) matrices, for example, [22–26]. However, an open access interval implementation of a singular value decomposition does not seem to exist. Note that the module VERTHINSVD [27] (`uivtx.cs.cas.cz/~rohn/matlab/`) is not available online at the moment. Moreover, the interval equivalents of matrices U and V are not especially meaningful, as explained in [28]. There is a possibility to (approximately) decompose a MIMO into several SISO links based, for example, on the GMD [29]. Moreover, the SVD decomposition can be performed using the CORDIC algorithm [30]. In our future work, we will consider these latter two possibilities to obtain interval versions of the weights $\sqrt{\xi_i}$ for (uncertain) channel coefficients.

3.3 Power Allocation Stage

In our current paper [31], we analysed the power allocation for frequency flat MIMO channels using a combined analytical-verified approach. If the Lagrange multipliers method is used to minimize the BER as the quality criterion, then the following cost function needs to be minimized:

$$\begin{aligned} J(p_1 \dots p_L, \lambda) = & \frac{2}{\sum\limits_{l=1}^{L} \log_2 M_l} \sum_{l=1}^{L} \left(1 - \frac{1}{\sqrt{M_l}}\right) \cdot \operatorname{erfc}\left(\frac{1}{2\sigma}\sqrt{\frac{3p_l \cdot \xi_l \cdot P_s}{L(M_l-1)}}\right) \\ & + \lambda \left(\sum_{l=1}^{L} p_l - L\right) \to \min \quad . \end{aligned} \tag{7}$$

Here, $p_1 \dots p_L$ are the power allocation parameters with which we modify the weights $\sqrt{\xi_i}$ from Eq. (6) in order to improve the overall BER. Moreover, M_l is the number of symbols for the layer $l \in \{1, \dots L\}$, λ is the Lagrange multiplier assigned to the constraint $\sum\limits_{l=1}^{L} p_l = L$ and P_s is the overall transmit power. This problem can be solved using interval methods with the help of the general-purpose global optimizer, for example, that implemented in the C-XSC Toolbox [32]. As a result, a verified upper bound on the lower bound for the BER can be computed. This leads to the overall verification degree of C2- for this part of the process. However, we could show that much better results (tighter interval bounds on $p_1 \dots p_L$) could be obtained by computing the derivatives analytically and solving the resulting system of nonlinear equations numerically using the corresponding verified solver provided, again, by the C-XSC Toolbox. The verification degree remains the same.

The verified upper bound on the BER (not on the lower bound of the BER) can be computed if the interference between the SISO layers of a MIMO link (cf. Section 3.2) is taken into account in the appropriate way. This is a part of our future work.

4 Estimating the Channel Matrix

In this section, we analyse the stage of the channel estimation in detail. First, the questionnaire leading to the assessment of the verification degree is reproduced. Next, the conventional procedure for channel estimation based on the least squares method is described. Finally, the results for the verified version of this subprocess are given.

4.1 Questionnaire for Channel Estimation

In this subsection, we reproduce the answers to the verification questionnaire for the stage of channel estimation. In practice, the algorithms can be implemented on an Application Specific Integrated Circuit (ASIC) or a Digital Signal Processor (DSP) which usually store data and carry out computations using a fractional/fixed point number format. We call it *practice mode* in the following. Alternatively, our goal can be testing the developed algorithms, in which case a PC simulation is sufficient (*simulation mode*).

The questions concern four distinct areas inside the subprocess: input data, models, algorithms and output data. They help to establish the current verification degree as C4- for the practice mode and C3 for the simulation mode. We improve the degree to C2- by using methods with result verification as described in Section 4.3.

4.1.1 Description of input data

The following initial information is necessary for this stage of the process: n_T, n_R, L_h (integer numbers), the pilot sequences S stored in a look-up table, and the received data R from an analog to digital converter.

The source of the initial received data can be an oscilloscope (for both the practice and the simulation mode) or a computer simulation; everything else is predefined.

Description form: The data is stored in a text file or in MATLAB script files. The pilot sequences' bits have the boolean data type. Symbols obtained by combining the corresponding bits can become integer or real or even complex depending on the application. The oscilloscope data is in Hierarchical Data Format (HDF5), the basic storage data type being fractional/fixed point (practice mode) or double IEEE 754 type (simulation mode).

Pre-selection: There is no data selection in this subprocess.

Accuracy: The pilot sequences are known exactly. The received data is provided in a discretised form by the ADC. Moreover, the receiver is disturbed by the Gaussian noise.

4.1.2 Description of models

The **formulas** are given in Section 3.1. The typical values of **parameters** are $n_{\mathrm{T}} > 1$, $n_{\mathrm{R}} > 1$ (2,3 or 4). The length of a pilot sequence L_{s} depends on the application. At the moment, there are no transmission standards using frequency selective channel estimation. In our example simulation in this section, we use the lengths 5, 33, 321. The sequences themselves can be devised analytically to fulfill the optimality condition $S^{\mathrm{H}}S = \alpha I$ [12] or can be generated randomly.

4.1.3 Description of algorithms

Type: The conventional algorithm described in Section 4.2 is of the numeric type for the solution of linear systems and of the stochastic type for the Monte Carlo simulation taking care of the noise.

Parallelization: The algorithm is parallelisable. For example, the channel coefficients can be computed in parallel for different MIMO outputs.

Architecture: ASIC or DSP in the practice mode. The current lab implementation uses a standard PC (simulation mode).

Operations: (Matrix) addition, multiplication. Additionally for arbitrary pilot sequences: transpose, complex conjugation.

Sub-algorithms: Least squares solution for overdetermined linear systems (for arbitrary pilot sequences).

There are no known nonnumerical factors the algorithm is especially **sensitive** to. The UML **description** of the algorithm is omitted for the sake of brevity.

4.1.4 Description of output

Data type: The output data (the estimated coefficients of H_i) are in the IEEE double format for the simulation mode, fixed point format for the practice mode; channel coefficients are unitless.

Accuracy: See the the mean squared error in Eqs. (8)-(9).

High accuracy: The estimations of the coefficients are the more accurate the higher the length of the pilot sequence L_{s}. However, this reduces the amount of transmitted useful information.

Failures: Aside from the possibility of the usual failures connected to the floating/fixed point numerics, the algorithm is robust.

Information exchange: The data is stored in a text file using the decimal number representation with 15 digits after the decimal point (simulation mode). In the practice mode, they stay in the memory in the usual fixed point format. There is no immediate need to plot the output data. The estimated coefficients are used afterwards for channel simulation.

4.2 Conventional Channel Estimation

In this section, the results achieved by the least squares broadband MIMO channel estimation algorithm are compared for different pilot sequence lengths L_s by means of a Monte Carlo simulation and a conventional floating-point implementation. For validation purposes, we choose a two tap frequency selective (2×2) MIMO channel, i.e., $L_h = 2$ and $n_R = n_T = 2$, with the known matrices H_0 and H_1 defined by

$$H_0 = \begin{pmatrix} \frac{4}{5} & \frac{12}{25} \\ \frac{2}{5} & \frac{16}{25} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.48 \\ 0.4 & 0.64 \end{pmatrix} \quad , \quad H_1 = \begin{pmatrix} \frac{2}{5} & \frac{6}{25} \\ \frac{1}{5} & \frac{8}{25} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.24 \\ 0.2 & 0.32 \end{pmatrix} \quad .$$

The resulting empirical probability density functions (PDFs) of the estimated channel taps are shown in Fig. 2 for three different pilot sequence lengths and σ corresponding to a fixed signal-to-noise ratio of 30 dB. The results show that a higher pilot sequence length reduces the variance of the estimated

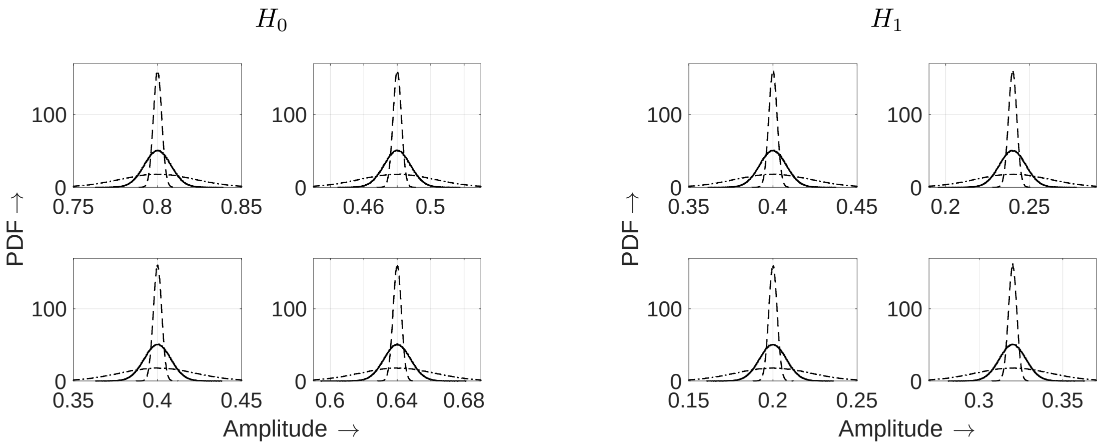

Figure 2: Empirical probability density functions of the estimated MIMO channel taps comparing different pilot sequence lengths — dash-dotted line: $L_s = 5$, solid line: $L_s = 33$, dashed line: $L_s = 321$

amplitudes. Specifically, the mean squared error (MSE) between the estimated channel taps in vector $\widehat{\mathbf{h}}_\nu$ and the original taps in vector $\mathbf{h}_\nu$ can be taken as a quality indicator as defined by

$$\mathrm{MSE} = \frac{1}{n_T\, n_R\, L_h} \cdot \mathrm{E}\left\{ \sum_{\nu=1}^{n_R} \left(\widehat{\mathbf{h}}_\nu - \mathbf{h}_\nu\right)^{\mathrm{H}} \cdot \left(\widehat{\mathbf{h}}_\nu - \mathbf{h}_\nu\right) \right\} \quad , \tag{8}$$

with $\mathrm{E}\{\,\cdot\,\}$ denoting the expectation function. Under additive white Gaussian noise disturbance and the use of optimal pilot sequences the MSE can be expressed analytically [12] by

$$\mathrm{MSE}_{\mathrm{LS}} = \frac{\sigma^2}{P_s} \cdot \frac{n_T}{L_s - (L_h - 1)} \quad . \tag{9}$$

This shows that choosing a longer pilot sequence reduces the estimation error, which confirms the Monte Carlo simulation results.

4.3 Computing Verified Bounds on the Channel Matrix

We implemented the whole channel estimation process using interval arithmetic and, in particular, the linear matrix equations solver provided by the C-XSC Toolbox. Instead of the Gaussian noise, we used the corresponding bounding intervals of $[-\sigma, \sigma]$, $[-2\sigma, 2\sigma]$ and $[-3\sigma, 3\sigma]$. Using such intervals means that we suppose that the density inside the bounds corresponds to the uniform distribution. The real numbers that we give are rounded to the third digit after the decimal point, if appropriate. The intervals are rounded outwards to the same number of digits. The length of the pilot sequence for the results shown below is $L_s = 5$. The channel matrices are the same as in the previous subsection, with the estimated values using Eq. (3) for the uncertainty of $\pm\sigma$ (± 0.0316) equal to

$$H_0 = \begin{pmatrix} [0.755, 0.845] & [0.435, 0.525] \\ [0.355, 0.445] & [0.595, 0.685] \end{pmatrix} \quad , \quad H_1 = \begin{pmatrix} [0.355, 0.445] & [0.195, 0.285] \\ [0.155, 0.245] & [0.275, 0.365] \end{pmatrix} \quad ,$$

$$\mathrm{mid}(H_0) = \begin{pmatrix} 0.8 & 0.48 \\ 0.4 & 0.64 \end{pmatrix} \quad , \quad \mathrm{mid}(H_1) = \begin{pmatrix} 0.4 & 0.24 \\ 0.2 & 0.32 \end{pmatrix} \quad ,$$

where the notation mid(·) means the matrix of the midpoints. The maximal width of the interval components of matrices H_0, H_1 is equal to 0.089443. Note that the midpoint matrices correspond to the real chosen channel. The estimated H_0, H_1 for the uncertainty of $\pm\sigma$ under the consideration of the optimality condition of the pilot sequences are

$$H_0 = \begin{pmatrix} [0.755, 0.845] & [0.435, 0.525] \\ [0.355, 0.445] & [0.595, 0.685] \end{pmatrix} , \qquad H_1 = \begin{pmatrix} [0.355, 0.445] & [0.195, 0.285] \\ [0.155, 0.245] & [0.275, 0.365] \end{pmatrix}$$

with the same midpoint matrices. That is, the estimated matrices are also numerically the same for both computational variants from Eqs. (3),(4). H_0, H_1 for the uncertainty of $\pm 2\sigma$ (± 0.063) are

$$\begin{aligned} H_0 &= \begin{pmatrix} [0.710, 0.890] & [0.390, 0.570] \\ [0.310, 0.490] & [0.310, 0.490] \end{pmatrix} , & H_1 &= \begin{pmatrix} [0.310, 0.490] & [0.150, 0.330] \\ [0.110, 0.290] & [0.230, 0.410] \end{pmatrix} , \\ \mathrm{mid}(H_0) &= \begin{pmatrix} 0.8 & 0.48 \\ 0.4 & 0.64 \end{pmatrix} , & \mathrm{mid}(H_1) &= \begin{pmatrix} 0.4 & 0.24 \\ 0.2 & 0.32 \end{pmatrix} , \end{aligned}$$

with the maximal width of 0.178885. Finally, H_0, H_1 for the uncertainty of $\pm 3\sigma$ (± 0.095) containing 99.7% of the possible values for the Gaussian noise are

$$H_0 = \begin{pmatrix} [0.665, 0.935] & [0.345, 0.615] \\ [0.265, 0.535] & [0.505, 0.775] \end{pmatrix} , \qquad H_1 = \begin{pmatrix} [0.265, 0.535] & [0.105, 0.375] \\ [0.065, 0.335] & [0.185, 0.455] \end{pmatrix} ,$$

again with the same midpoint matrices. The maximal width is 0.268328.

The estimated H_0, H_1 for $L_s = 33$ and $L_s = 321$ are almost exactly the same. That is, the length of the pilot sequence does not play any role for the quality of the obtained intervals and can be chosen to be minimal ($L_s = 5$). The result is not surprising in this setting since the artificially generated data for the received vectors in the matrix R are obtained not by adding a specific value of the noise to the coefficients of H_i (as in Section 4.2) but by considering an interval enclosing relevant values which is always the same for a given σ. That is, the optimal solution of the overdetermined system in Eq. (2) in the sense of the least squares method is the same independently of the pilot sequence length. However, we expect similar behaviour in the real life settings using actual measurements for the responses R.

In Figure 3, the relationship between the PDFs obtained with the Monte Carlo simulation and the interval enclosure containing 99.7% of the possible noise values is exemplified for the coefficient in the first row and the first column of the channel matrix H_0. The conservativeness of the interval enclosure in the figure is explained by the width of the input uncertainty which is of order $2 \cdot 10^{-1}$ ($\pm 3\sigma$). The result has the width of the same order of magnitude which is quite good for interval methods considering that the task is linear.

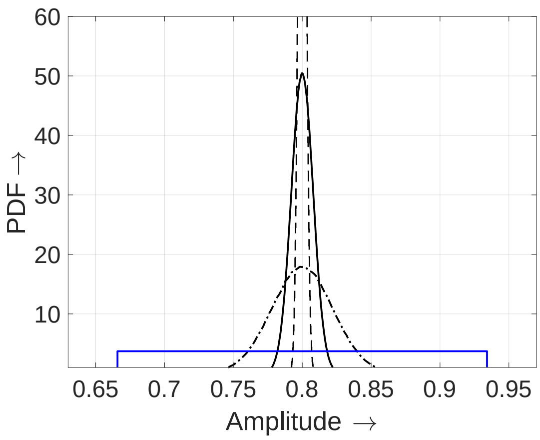

Figure 3: Channel coefficient $H_0(1,1)$: Empirical PDFs for pilot sequence lengths $L_s = 5$ (dash-dotted line), $L_s = 33$ (solid), $L_s = 321$ (dashed) and the interval enclosure (blue line) for $\pm 3\sigma$

5 Conclusions

In this paper, we analysed the task of digital MIMO communications using V&V techniques. We described how to improve the verification degree of the three possible stages of this process with a special focus on the channel estimation for frequency selective MIMO systems. We could show that verified techniques could help to reduce the pilot sequences' lengths at the same time taking care of numerical errors and enclosing the additive uncertainty. As future work, the proposed interval estimation procedure should be tested for real life systems with unknown channels in the simulation mode. Our long-term goal is the implementation of interval procedures for the practice mode (i.e., on ASIC chips).

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