P-solutions of linear and nonlinear interval parametric systems and applications

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Abstract. As is well known, linear interval parametric (LIP) systems constitute an adequate mathematic model for various engineering problems involving uncertain parameters given in intervals. Presently, the following “interval solutions” of a LIP system are mostly considered: (i) outer interval (OI) solution, (ii) inner estimation of the hull (IEH) solution and (iii) interval hull (IH) solution. A new type of solution of a LIP system, called parameterized or *p*-solution, has been recently introduced. The new solution has a number of useful properties such as direct determination of the OI and IEH solutions, computing two-sided bounds on the ends of the IH solution. Combined with a constraint satisfaction technique, it permits to develop a novel framework for solving various global optimization problems where the constraint is a LIP system. The objective of the present paper is to overview the concept of the *p*-solution and its applications in solving various engineering problems. These include: worst-case tolerance analysis of electric circuits, determining the power consumption range in direct or alternating current circuits, truss analysis of mechanical structures, eigenvalue range determination for parametric matrices. It is shown that the new methods employing *p*-solutions have improved computational efficiency as compared to other known methods.

1. Introduction

Mathematic models including parametric equations are encountered in various fields of science such as engineering, physics, computer science, technologies, economics etc. In realistic parametric models, the parameters involved are not known exactly. Presently, various approaches to tackling the inherent parameter uncertainty are in use: probabilistic, fuzzy sets, interval analysis etc. In this paper, we employ the interval analysis approach according to which the uncertain parameters take on their values within prescribed intervals. Therefore, some facts from interval analysis needed in the sequel are given in Sect. 1.1.

In this paper, we consider systems of linear and nonlinear parametric equations. More specifically, we are interested in the determination and application of the so-called parameterized solution of such systems. The formulation of that problem will be presented in Sect. 1.2.

1.1. Preliminaries [1,2]

We will use boldface to denote interval quantities, underscores to denote lower bounds and overscores to denote upper bounds. Scalars and vectors will, generally, be denoted by lower case letters while matrices will be denoted by upper case. An interval $\boldsymbol{a} = \{a : \underline{a} \le a \le \overline{a},\ \underline{a} \le \overline{a}\}$ can be denoted either in its lower-upper-end form

$$\boldsymbol{a} = [\underline{a},\ \overline{a}] \tag{1.1a}$$

or in a center-radius form

$$\boldsymbol{a} = \breve{a} + [-\hat{a}, \hat{a}] = \breve{a} + \hat{a}[-1, 1], \quad \breve{a} = (\underline{a} + \overline{a})/2,\ \hat{a} = (\overline{a} - \underline{a})/2. \tag{1.1b}$$

The same notations will be used for the components $\boldsymbol{x}_i$ of an interval vectors $\boldsymbol{x} = (\boldsymbol{x}_1, ..., \boldsymbol{x}_n)$ of size *n* where $\boldsymbol{x}_i$ are independent intervals. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are interval vectors, the relations $=$, $\le$, $\subset$ etc. are meant component-wise. It should be stressed that geometrically an interval vector $\boldsymbol{x}$ is a box in R^n.

Bounds on the range of a function

Let $f(x)$, $x \in \boldsymbol{x}$ be an *explicitly defined* nonlinear function $f : \boldsymbol{x} \subset R^n \to R$. The range $f(\boldsymbol{x})$ of *f* is the interval $f(\boldsymbol{x}) = \{f(x) : x \in \boldsymbol{x}\}$. Using $\boldsymbol{x}$ and interval arithmetic (IA) [1], we can compute the so-called interval extension $\boldsymbol{f}(\boldsymbol{x})$ (which is also an interval) of *f*, having the inclusion property

$$f(\boldsymbol{x}) \subset \boldsymbol{f}(\boldsymbol{x}). \tag{1.2}$$

There is an alternative approach to computing an interval extension of $f(x)$ over $\boldsymbol{x}$ which is based on the use of the so-called affine forms [2]. An affine form $\langle x \rangle$ is defined as

$$\langle x \rangle = \breve{x} + \sum_{j=1}^{n} x_j \xi_j, \quad \xi_j \in \boldsymbol{\xi}_j = [-1, 1] \tag{1.3}$$

where $\breve{x}$ and x_j are given scalars. Using affine arithmetic (AA) and Chebyshev approximations of univariate sub-expressions in $f(x)$ [2], we compute the affine form $\langle f \rangle$ associated with $f(\boldsymbol{x})$:

$$\langle f \rangle = \breve{f} + \sum_{j=1}^{n} f_j \xi_j + \sum_{j=n+1}^{n'} f_j \xi_j,\ \xi_j \in \boldsymbol{\xi}_j. \tag{1.4}$$

The first summation keeps track of first-order correlations between the components x_j in $f(x)$; the additional terms in the second summation result from nonlinear arithmetic operations or Chebyshev/ min-range approximations. The smallest interval containing $\langle f \rangle$ is denoted $\langle \boldsymbol{f} \rangle$ and is given as

$$\langle \boldsymbol{f} \rangle = \breve{f} + \sum_{j=1}^{n'} |f_j|. \tag{1.5}$$

In the framework of the AA approach, $\langle \boldsymbol{f} \rangle$ is used rather than $\boldsymbol{f}(\boldsymbol{x})$ to assess the range $f(x)$ since the following inclusion is valid

$$f(\boldsymbol{x}) \subset \langle \boldsymbol{f} \rangle \tag{1.6}$$

and most often

$$\langle \boldsymbol{f} \rangle \subset \boldsymbol{f}(\boldsymbol{x}). \tag{1.7a}$$

Unfortunately, the above relation is not guaranteed and sometimes the interval extension is a better option, i.e.

$$\boldsymbol{f}(\boldsymbol{x}) \subset \langle \boldsymbol{f} \rangle. \tag{1.7b}$$

Remark 1.1. In view of (1.7) the following simple strategy has been adopted in [2]: compute both $\boldsymbol{f}(\boldsymbol{x})$ and $\langle \boldsymbol{z} \rangle$ and choose the narrowest interval for subsequent computations.

Linear interval parametric (LIP) enclosures of a function

Such enclosures were proposed for outward linear approximation of a nonlinear function *f*(*x*), $x \in \boldsymbol{x}$ in the framework of solving systems of nonlinear equations [3-6]. A LIP enclosure has the form

$$\boldsymbol{f}(x) = \sum_{j=1}^{n} a_j x_j + \boldsymbol{a},\ x_j \in \boldsymbol{x}_j \tag{1.8}$$

where a_j are real numbers whereas $\boldsymbol{a}$ is an interval. It is seen that (1.8) is a multivalued function so it can be viewed geometrically as a "sandwich" region in R^n. It should be stressed that the latter region is a much better approximation of the graph of the function *f*(*x*) in $\boldsymbol{x}$ than the intervals $\boldsymbol{f}(\boldsymbol{x})$ or $\langle \boldsymbol{f} \rangle$ which only bound the range of *f*(*x*). The LIP enclosure has the property

$$f(\boldsymbol{x}) \subset \boldsymbol{f}(x),\ x \in \boldsymbol{x} \tag{1.8a}$$

An automatic procedure for determining a_j and a was suggested in [4].

In our approach, we prefer to use AA to determining a linear enclosure of $f(x)$ in $\boldsymbol{x}$ since AA has been implemented as a toolbox (called *affari*) in Intlab [7] (a toolbox in Matlab). However, we modify (1.4) as follows: the second summation is lumped into an interval of radius

$$f_{n+1} = \sum_{j=n+1}^{n'} |f_j|$$

so (1.4) is rewritten in the form

$$\langle \boldsymbol{f} \rangle = \breve{f} + \sum_{j=1}^{n} f_j \xi_j + [-f_{n+1}, f_{n+1}],\ \xi_j \in \boldsymbol{\xi}_j\,. \tag{1.9}$$

1.2. Problem statement

We are concerned with the following parametric systems of decreasing generality and complexity. The first is a nonlinear interval parametric (NLIP) system

$$f_i(x_1,..x_n; p_1,..p_m) = 0,\ \ p_j \in \boldsymbol{p}_j,\ i = 1,..,n$$

where x_j are the variables and p_j are the parameters that vary within given intervals $\boldsymbol{p}_j$. The above system is written compactly in the form

$$f(x,p) = 0\,,\quad p \in \boldsymbol{p} \tag{1.10}$$

where the function $f(x,p)$ is $f: R^n \times R^m \to R^n$, *n* and *m* being the size of the variable vector *x* and parameter vector *y*, respectively. The second system is linear in *x* so (1.10) becomes

$$f(x,p) = A(p)x - b(p) = 0\,,\ p \in \boldsymbol{p}\,. \tag{1.11}$$

The elements of $A(p)$ and $b(p)$ are, however, nonlinear functions of *p*, i.e.

$$a_{ij}(p) = a_{ij}(p_1,...,p_m),\quad b_i(p) = b_i(p_1,...,p_m)\,,\ \ p \in \boldsymbol{p}\,. \tag{1.11a}$$

Such systems will be referred to as nonlinear parametric dependences (NLPD) systems). When the functions in (1.11a) are (affine) linear functions

$$a_{ij}(p) = \alpha_{ij} + \sum_{\mu=1}^{m} a_{ij\mu} p_\mu\,,\quad b_i(p) = \beta_i + \sum_{\mu=1}^{m} \beta_{i\mu} p_\mu \tag{1.11b}$$

we have the third type of parametric system which is defined by (1.11) and (1.11b). Such systems will be referred to as linear parametric dependence (LPD) systems.
We need the concept of the solution set Σ of (1.10) or (1.11) defined as follows

$$\Sigma = \{x : f(x, p) = 0,\ p \in \boldsymbol{p}\}. \tag{1.12}$$

In view of (1.12) the following "interval solutions" to (1.10) or (1.11) can be considered: (i) interval hull (IH) solution $\boldsymbol{x}^*$: the smallest interval vector containing Σ; (ii) outer interval (OI) solution $\boldsymbol{x}$: any interval vector enclosing $\boldsymbol{x}^*$, i.e. $\boldsymbol{x}^* \subseteq \boldsymbol{x}$ and (iii) inner estimation of the hull (IEH) solution $\boldsymbol{\xi}$: an interval vector such that $\boldsymbol{\xi} \subseteq \boldsymbol{x}^*$.

Various methods for determining the above solutions when they are related to the linear systems (1.11) have been suggested in the literature (see the introductory sections of [8, 9]). An approach to computing the OI solution $\boldsymbol{x}$ in the general nonlinear system (1.10) is known [1] which is based on applying the parametric interval Newton method to (1.10).

It should be stressed that all three solutions are in the form of interval vectors.

Recently, a new type of solution to LPD systems (1.11), (1.11b) has been introduced [8] which is of the following parametric form

$$\boldsymbol{x}(p) = Lp + \boldsymbol{a},\quad p \in \boldsymbol{p} \tag{1.13}$$

where L is a real $n \times m$ matrix whereas $\boldsymbol{a}$ is an interval vector. It is called a parameterized solution (*p*-solution) of (1.10). The *p*-solution is an outward linear approximation of Σ. It has been shown that $\boldsymbol{x}(p)$ is a much better approximation of Σ than $\boldsymbol{x}$.

Iterative methods for determining $\boldsymbol{x}(p)$ were suggested for the case of LP dependences in [8-10]; a simple direct method was proposed in [11]. Also the case of NLP dependences was considered in [12, 13]. The concept of a *p*-solution was extended in [14] to the so-called AE-systems with LP dependences.

The *p*-solution has a number of useful properties: it permits determining an outer interval solution $\boldsymbol{x}$, an inner estimation $\boldsymbol{\xi}$ of the hull solution as well intervals containing the endpoints $\underline{x}_i^*$ and $\overline{x}_i^*$ of each component $\boldsymbol{x}_i^*$ of the interval hull solution x^* related to (1.10). The main advantage of $\boldsymbol{x}(p)$, however, is that it can be laid as the basis of a new unified approach for solving the following optimization problem: find the global minimum

$$f_0^* = \min f_0(x, p) \tag{1.14}$$

subject to the constraint (1.11) where $f_0(x, p)$ is, in the general case, a nonlinear function.

The objective of the present paper is to overview the concept of the *p*-solution and its applications. It will outline the properties of the *p*-solutions, the available methods for their computation as well as the potential of their use in solving various engineering problems.

The paper is organized as follows. In Sect. 2 it is shown that the NLPS (1.10) can be transformed into a LPS (1.11). The parameterised form solution $\boldsymbol{x}(p)$ of (1.11) and its properties are introduced in the first subsection of Section 3. In the next subsection, a direct method for determining $\boldsymbol{x}(p)$ (i.e. for computing L and $\boldsymbol{a}$) is presented for the special case of LPD systems (1.11), (1.11b). Also an iterative method for computing $\boldsymbol{x}(p)$ related to the NLPS (1.10) is suggested. The potential of the new approach for solving global minimization problems of the type (1.14) is illustrated in Section 4. Several applications in the fields of electrical, civil and mechanical engineering such as worst-case tolerance analysis of linear electric circuits, determining the active and reactive power range in direct or alternating current circuits, truss analysis of mechanical structures, eigenvalue range determination for parametric matrices are given there. It is shown that the new methods employing *p*-solutions have improved computational efficiency and, hence, a larger radius of applicability (capacity to solve problems of larger parameter uncertainties) as compared to other known methods. The paper ends up with several concluding remarks.

2. Linear parametric models

In this section, it is shown how the NLPS (1.10) can be transformed into a LPS (1.11) [15].

2.1. Systems of nonlinear (nonparametric) equations

We first show how a nonlinear (nonparametric) system can be modeled by a LPS.

The problem considered is: solve globally

$$f(x) = 0\,, \tag{2.1a}$$

$$x \in \boldsymbol{x}^0 \subset R^n\,, \tag{2.1b}$$

that is find all zeros of $f(x)$ in $\boldsymbol{x}^0$. In the framework of the interval analysis approach [1], the most popular interval method for global solution of nonlinear systems is the well-known interval Newton method (or its versions). It is an iterative method using an interval extension $\boldsymbol{J}(\boldsymbol{x})$ of the Jacobean matrix (or some modifications) for the currant box $\boldsymbol{x}$ belonging to $\boldsymbol{x}^0$ and related to a given iteration. The following interval system

$$\boldsymbol{A}(\boldsymbol{x})\, y = f(x) \tag{2.2}$$

is to be solved at each iteration where $\boldsymbol{A}(\boldsymbol{x})$ is an interval matrix (standing for the interval Jacobean matrix, interval slope matrix, the Hansen-Sengupta operator or some other modification) while x and hence $f(x)$ is a real vector.

An alternative approach was suggested in [3-5] using the following enclosure

$$f(y) \in A(\boldsymbol{x})\, x + \boldsymbol{b}(\boldsymbol{x})\,, \quad x \in \boldsymbol{x} \tag{2.3}$$

where $A(x)$ is a real matrix while $\boldsymbol{b}(\boldsymbol{x})$ is an interval vector. The right side in (2.3)

$$L(\boldsymbol{x}) = A(\boldsymbol{x})\, x + \boldsymbol{b}(\boldsymbol{x})\,, \; x \in \boldsymbol{x} \tag{2.4}$$

is known as a *linear interval approximation* (LIA). It has the important property

$$f(x) \in L(\boldsymbol{x}) \tag{2.4a}$$

In this case solving (2.1) reduces to repeatedly solving the interval system

$$A(\boldsymbol{x})\, y = -\boldsymbol{b}(\boldsymbol{x}) \tag{2.5}$$

where, unlike (2.2), $A(\boldsymbol{x})$ is now a real matrix. This determines the better performance of the LIA approach as compared to the standard approach based on (2.2).

A new approach for global nonlinear analysis was suggested in [15]. It is based on various alternative approximations of f in $\boldsymbol{x}$ which are of linear parametric form. Now, we obtain a linear parametric system

$$A(p)(y - x) = -f(x)\,, \quad p \in \boldsymbol{p} \tag{2.6a}$$

or, equivalently

$$A(p)\, z = b(x)\,, \quad p \in \boldsymbol{p} \tag{2.6b}$$

where $A(p)$ is a parametric matrix while x is fixed. It has been shown that (2.6) is a better way to bound the solutions of (2.1) than the interval Newton method. System (2.6) will be referred to as a *linear parametric model* (LPM) of $f(x)$ in $\boldsymbol{x}$.

In the following subsections we present several specific implementations for the LPM.

2.1.1. Linear parametric model using slopes. It is known that the slope matrix leads to better approximation of nonlinear parametric functions than the Jacobian matrix. That is why the former matrix was used in [15] to construct a LPM.

The novel approach is based on the use of the slope matrix $S(y,x)$ and the equality

$$f(y) = f(x) + S(y, x)\,(y - x)\,, \tag{2.7}$$

where y and x have some fixed values (typically, x is a known solution x^0 of (2.1)). Now the components y_k of y are "freed"and considered as components of a parameter vector p, i.e.

$$p = (y_1,...,y_n) \in \boldsymbol{x} = (\boldsymbol{x}_1,...,\boldsymbol{x}_n)\,. \tag{2.8}$$

Let

$$a_{ij}(p) = S_{ij}(p,x) \tag{2.9}$$

be the entries of the parametric matrix $A(p)$. On account of (2.7) to (2.9)

$$y \in f(x) + A(p)(y - x), \quad p \in \boldsymbol{x}\,. \tag{2.10}$$

The right-hand side of (2.10) is the *novel linear parametric model* (LPM) of f in $\boldsymbol{x}$. If y is a zero of f, then

$$A(p)(y - x) = -f(x)\,, \quad p \in \boldsymbol{x} \tag{2.11a}$$

or, equivalently

$$A(p)\,z = b(x)\,, \quad p \in \boldsymbol{x}\,. \tag{2.11b}$$

Thus, using the novel approximation, the linear parametric model (2.11) is obtained. The latter model, defined by (2.9) to (2.11b), is a better way than (2.2) to bound the solutions of (2.1) [15]. Indeed, introduce the solution sets

$$S_J = \{z: \ \ J\,z = b\,, \ \ J \in \boldsymbol{J}(\boldsymbol{x})\}\,, \tag{2.12}$$

$$S_p = \{z: \ \ A(p)\,z = b\,, \ \ p \in \boldsymbol{x}\}\,. \tag{2.13}$$

It is seen that while $\boldsymbol{J}(\boldsymbol{x})$ depends on n^2 independent entries, there are only n independent elements in $A(p)$. Moreover, the methods for enclosing S_p account for the interdependencies between the components of $A(p)$. Thus, it follows from (2.12) and (2.13) that

$$S_p \subset S_J\,. \tag{2.14}$$

If Z^J_{out} and Z^p_{out} denote some outer interval solution of (2.2) and (2.11b), respectively, then we can expect that also

$$Z^p_{\text{out}} \subset Z^J_{\text{out}}\,. \tag{2.15}$$

2.1.2. LPA using the Hansen-Sengupta operator in parametric form. The approach of § 2.1.1 is applicable only if the slope matrix $S(y, x)$ is available in analytical form. If this is not the case, then the Jacobian matrix in parametric form $J(p)$ can be used as suggested in [15]. Thus, (2.9) is replaced with

$$a_{ij}(p) = J_{ij}(p_1,,,p_n)\,,\quad p_i \in \boldsymbol{p}_i = \boldsymbol{x}_i\,,\ i = 1,,,n\,. \tag{2.16}$$

It is seen that each element $a_{ij}(p)$ depends on all n parameters p_i. A better LPA is suggested here which is based on the Hansen-Sengupta operator [1]. In its standard (nonparametric) form, it encloses each function $f_i(y)$ by the following expression

$$f_i(y) \in f_i(x) + \sum_{j=1}^{n}\left(y_j - x_j\right) g_{ij}\left(\boldsymbol{x}_1,..,\boldsymbol{x}_j;x_{j+1},..,x_n\right). \tag{2.17}$$

We now write (2.17) in parametric form

$$f_i(y) \in f_i(x) + \sum_{j=1}^{n}\left(y_j - x_j\right) g_{ij}\left(p_1,..,p_j;x_{j+1},..,x_n\right),\ p_1 \in \boldsymbol{x}_1,..,p_j \in \boldsymbol{x}_j\,. \tag{2.18}$$

Hence, using the Hansen-Sengupta operator, we have to replace (2.16) with

$$a_{ij}(p) = g_{ij}(p_1,,,p_j)\,,\quad p_k \in \boldsymbol{p}_k = \boldsymbol{x}_k\,,\ k = 1,..,j\,. \tag{2.19}$$

It is seen that, unlike (2.16) where all n parameters are intervals, now a fraction $n(n-1)/2$ in (18) are real parameters. This determines the better performance of the Hansen-Sengupta LPA as compared to the Jacobian LPA.

2.2. Systems of nonlinear parametric equations

We now extend some of the ideas considered earlier to the parametric case

$$f(x,p) = 0\,, \tag{2.20a}$$

$$p \in \boldsymbol{p} \subset R^m\,, \tag{2.20b}$$

$$x \in \boldsymbol{x}^0 \subset R^n\,. \tag{2.20c}$$

The solution set of (20a), (20b) (as defined in (1.12)) is the set

$$\Sigma := \{x : f(x,p) = 0,\ p \in \boldsymbol{p}\} \tag{2.21}$$

which defines the IH solution $\boldsymbol{x}^*$ and the OI solution $\boldsymbol{x}'$.

2.2.1. A basic problem. A basic problem is to determine $\boldsymbol{x}'$ for a given f and p. Finding $\boldsymbol{x}'$ has been considered as a sensitivity analysis problem associated with (2.20a), (2.20b) (e.g., [1]). Various methods and algorithms are based on the parameterized versions of the Newton method and its variants. Now

$$f(y,p) \in f(x) + \boldsymbol{J}(\boldsymbol{x},\boldsymbol{p})(y - x) \tag{2.22}$$

is used which becomes

$$\boldsymbol{J}(\boldsymbol{x},\boldsymbol{p})(y - x) = -f(x,p) \tag{2.23}$$

if (y, p) is a zero of f. Another alternative idea is based on the use of the LAI (2.4) applied to the function $f(u)$ when $u = (x,p)$.

2.2.2. New linear parametric model (LPM) approach. In what follows, we extend the model from 2.1.1 and 2.1.2 to the parametric system (2.20). In the case of the parametric equation (2.20a), for a fixed p formula (2.7) becomes

$$f(y,p) = f(x,p) + S(y,\, x,p)\ (y-x) \tag{2.24}$$

where most often x is the centre of $\boldsymbol{x}$. Once again, the components y_k of y are "freed" to take on values in $\boldsymbol{x}$ [15]. Thus, we introduce the additional parameter vector

$$q = (y_1,\ldots,y_n) \in \boldsymbol{x} = (\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\,. \tag{2.25a}$$

and let

$$a_{ij}(q,p) = S_{ij}(q,x,p,) \tag{2.25b}$$

be the entries of the parametric matrix $A(q, p)$, $q \in \boldsymbol{x}$, $p \in \boldsymbol{p}$ (x is fixed). For each $p \in \boldsymbol{p}$

$$f(y,p) \in f(x) + A(q,p)(y-x), \quad q \in \boldsymbol{x}, \quad p \in \boldsymbol{p}\,. \tag{2.26}$$

If (y,p) is a zero of f

$$A(q,p)z = b(p), \ q \in \boldsymbol{x}, \ p \in \boldsymbol{p} \tag{2.27}$$

where $z = y - x$ and $b(p) = -f(x,p)$. The linear parametric system (2.27) is the new LPM model suggested to tackle the problem of obtaining an outer approximation $\boldsymbol{y}$ of the solution set Σ. Indeed, consider the sets

$$S_{JP} = \{z: \ \ J\,z = b\,, \quad J \in \boldsymbol{J}(\boldsymbol{x},\boldsymbol{p}\}\,, \tag{2.28}$$

$$S_{pq} = \{z: \ \ A(q,p)\,z = b(p)\,, \quad q \in \boldsymbol{x},\, p \in \boldsymbol{p}\}\,. \tag{2.29}$$

Clearly,

$$S_{pq} \subset S_{JP} \tag{2.30}$$

($\boldsymbol{J}(\boldsymbol{x},\boldsymbol{p})$ has n^2 independent entries, each being an interval extension of the function $J(x,p)$ of $n+m$ arguments, while there are only $n+m$ independent elements in $A(q,p)$ and m independent elements in $b(p)$.) Hence we can expect (2.27) to be a better model than (2.2).

If the slopes $S_{ij}(q,x,p)$ cannot be found in analytical form, then they should be replaced with the components $g_{ij}(q,p)$. In that case, the elements of the parametric matrix $A(q,p)$ are given not by (2.26) but as follows $a_{ij}(q,p) = g_{ij}(q_1,..,q_j;p)$.

Remark 2.1. It should be noted that these LPMs have not been used yet and no numerical evidence about their efficiency is available. One exception is [16] where the LPM using slopes (Sect. 2.1.1) is studied. As an example, a system of quadratic functions has been solved. The results obtained confirm the superiority of the LPM approach over the standard method used as regards applicability radius in computing outer solutions, contracting effect, checking absence of solution, checking uniqueness of a solution.

3. P-solutions

3.1. Properties

We assume that a p-solution to the nonlinear parametric system (1.10) can be determined. (A method for computing such a solution will be suggested in Sect. 3.3). The properties of the p-solution in the general case of system (1.10) are similar to those related to the linear parametric system (1.11), (1.11b) [8].

Lemma 3.1. Let $x(p)$ be a p-solution of system (1.10). Then the interval hull $\boldsymbol{x}(\boldsymbol{p})$ of $x(p)$ is an outer interval solution $\boldsymbol{x}$ of (1.10), i.e. $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p})$.

Lemma 3.2. The ith component $\boldsymbol{x}_i^*$ of the IH solution $\boldsymbol{x}^*$ to (2.10) is contained in the ith component $x_i(\boldsymbol{p})$ of the range $\boldsymbol{x}(\boldsymbol{p})$, i.e. $\boldsymbol{x}_i^* \subset \boldsymbol{x}_i(\boldsymbol{p})$.

The lemma can be improved as follows. Consider the ith component $\boldsymbol{x}_i(p)$ of $\boldsymbol{x}(p)$

$$\boldsymbol{x}_i(p) = \hat{x}_i + L_i p + \hat{x}_i[-1,1], \quad p \in \boldsymbol{p}. \tag{3.1}$$

(L_i denote the ith row of L). Compute the ends

$$\underline{x}_i = \breve{x}_i - \sum_{ij} |L_{ij}| - \hat{x}_i, \quad \overline{x}_i = \breve{x}_i + \sum_{ij} |L_{ij}| + \hat{x}_i \tag{3.2}$$

and the hull

$$\boldsymbol{x}_i = \boldsymbol{x}_i(\boldsymbol{p}) = [\underline{x}_i, \overline{x}_i]. \tag{3.3}$$

We introduce the following two intervals:

$$\boldsymbol{e}_i^{(l)} = [\underline{x}_i, \underline{x}_i + 2\hat{x}_i], \quad \boldsymbol{e}_i^{(u)} = [\overline{x}_i, \overline{x} - 2\hat{x}_i]. \tag{3.4}$$

The following result is valid.

Theorem 3.5. Let $\boldsymbol{e}_i^{(l)}$ and $\boldsymbol{e}_i^{(u)}$ be the intervals defined by (3.4); also let $\underline{x}_i^*$ and $\overline{x}_i^*$ be the endpoints of $\boldsymbol{x}_i^*$. Then

$$\underline{x}_i^* \in \boldsymbol{e}_i^{(l)}, \quad \overline{x}_i^* \in \boldsymbol{e}_i^{(u)}. \tag{3.5}$$

The proof is similar to that of Theorem 1 in [8].

Corollary 3.2. Introduce the interval

$$\xi_i = \begin{cases} [\overline{e}_i^{(l)}, \underline{e}_i^{(u)}], & if\ \overline{e}_i^{(l)} < \underline{e}_i^{(u)} \\ \text{empty interval}, & otherwise \end{cases} \tag{3.6}$$

Then ξ_i determines the ith component of the IEH solution of (1.10).

Also, we give a new property (not mentioned in [8]): computation of tight enclosures for $f_i(x,y)$, $y \in \boldsymbol{y}$ in (1.10). To show this possibility, we replace y_i and x_i into the ith function in (1.10) by their AA forms and carry out the AA operations involved. Thus, we obtain a LIP form containing only the p variable:

$$\boldsymbol{f}_i(p) = \breve{f}_i + L_i^f p + \hat{f}_i[-1,1], \quad p \in \boldsymbol{p}. \tag{3.7a}$$

Let $\boldsymbol{f}_i(\boldsymbol{x},\boldsymbol{y})$ be the interval extension of $f_i(x,y)$, $x \in \boldsymbol{x}$, $y \in \boldsymbol{y}$; also let $\langle \boldsymbol{f}_i \rangle$ denote the interval hull of the affine form obtained by direct application of AA operations in $f_i(x,y)$ using (1.4). It is natural to expect that the interval hull $\boldsymbol{f}_i(\boldsymbol{p})$ of $\boldsymbol{f}_i(p)$ is narrower than both $\boldsymbol{f}_i(\boldsymbol{x},\boldsymbol{y})$ and $\langle \boldsymbol{f}_i \rangle$, i.e.

$$\boldsymbol{f}_i(\boldsymbol{p}) \subset \boldsymbol{f}_i(\boldsymbol{x},\boldsymbol{y}), \quad \boldsymbol{f}_i(\boldsymbol{p}) \subset \langle \boldsymbol{f}_i \rangle. \tag{3.7b}$$

This will be shown numerically in the next section.

Next, we present two methods for determining a p-solution: the known direct method [11] in Subsection 3.2 and a new iterative method in Subsection 3.3.

3.2. Direct method

In the case of LPD or NLPD systems (1.11), (1.11a) or (1.11), (1.11b), a *p*-solution can be computed using some of the known methods [8-13]. If the system is LPD, then the direct method of [11] could be the first choice (it is also used as an initial step in an iterative method [10] of better accuracy). For illustrative purposes, we present the latter method in the next subsection.

3.2.1. Computational scheme. Let the LPD system be given in the form

$$A(p)x = a(p), \quad p \in \boldsymbol{p} = [-\hat{p}, \hat{p}] \tag{3.8a}$$

$$A(p) = \breve{A} + \sum_{\mu=1}^{m} A^{(\mu)} p_\mu, \qquad a(p) = \breve{a} + A^0 p \tag{3.8b}$$

where $\breve{A}$, $A^{(\mu)}$ are matrices of size $n \times n$ and A^0 is $n \times m$.

The direct method for determining $\boldsymbol{x}(p)$ of (3.8) comprises the following steps.

1. First, we assume (temporarily) that $A(\boldsymbol{p})$ is regular so $R = \breve{A}^{-1}$ exists. Using *R*, (3.8) is transformed equivalently to

$$B(p)y = b(p), \quad p \in [-\hat{p}, \hat{p}], \tag{3.9}$$

$$B(p) = I + \sum_{\mu} B^{(\mu)} p_\mu, \quad B^{(\mu)} = RA^{(\mu)}, \quad p_\mu \in [-1, 1], \tag{3.9a}$$

$$b(p) = B^0 p, \quad B^0 = R(A^0 + C^0), \quad C^0_{:\mu} = A^{(\mu)} \breve{x} \tag{3.9b}$$

($C^0_{:\mu}$ is the μ th column of C^0). It is seen from (3.8), (3.9) that the original problem of finding the *p*-solution $\boldsymbol{x}(p)$ has been reduced to finding the *p*-solution $\boldsymbol{y}(p)$ of (3.9).

2. Now an interval (nonparametric) matrix $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{p})$ is introduced using (3.9a):

$$\boldsymbol{B} = I + \sum_{\mu} B^{(\mu)} \boldsymbol{p}_\mu = I + \Delta[-1, 1], \quad \Delta = \sum_{\mu} \left| B^{(\mu)} \right|. \tag{3.10}$$

Next, (3.9) is enclosed in $\boldsymbol{p}$ by the following "mixed type" system

$$Bu = b(p), \quad B \in \boldsymbol{B}, \quad p \in \boldsymbol{p}. \tag{3.11}$$

It is assumed that

$$\sigma(\Delta) < 1 \tag{3.12}$$

so *B* is regular.

3. Equation (3.11) is written as

$$u(p) = B^{-1} B^0 p, \quad B \in \boldsymbol{B}, \quad p \in \boldsymbol{p} \tag{3.13a}$$

and is replaced with

$$v(p) = \boldsymbol{H} B^0 p, \quad \boldsymbol{H} = \boldsymbol{B}^{-1}, \quad p \in \boldsymbol{p}. \tag{3.13b}$$

Due to (3.12), the matrix $\boldsymbol{H}$ can be computed [11]

$$\boldsymbol{H} = \breve{H} + \hat{H}[-1, 1] \tag{3.14}$$

where $\breve{H}$ is a diagonal matrix whose non-zero elements are all positive ($\breve{H}_{ii} > 0$).

4. In view of (3.14), equation (3.13) is rewritten in the form

$$\boldsymbol{v}(p) = \breve{H}B^0 p + \left(\hat{H}\left|B^0\right|\hat{p}\right)[-1,1] \tag{3.15}$$

hence

$$\boldsymbol{v}(p) = Lp + \boldsymbol{s}, \qquad \boldsymbol{s} = [-s, s], \tag{3.16a}$$

$$L = \breve{H}B^0, \quad s = \hat{H}\left|B^0\right|\hat{p}. \tag{3.16b}$$

and it is seen that L is real $n \times m$ matrix while $\boldsymbol{s} = [-s, s]$ is a symmetric interval vector.

5. Finally, it is shown that the p-solution sought is given by

$$\boldsymbol{x}(p) = \breve{x} + Lp + \boldsymbol{s}, \quad p \in \boldsymbol{p} \tag{3.17}$$

where $\breve{x}$ is the solution of $\breve{A}x = \breve{a}$.

Theorem 3.2. Let $\breve{A}$ in (3.8b) be nonsingular. Assume that condition (3.12) is fulfilled. Then
(i) $A(\boldsymbol{p})$ is a regular parametric matrix;
(ii) the p-solution $\boldsymbol{x}(p)$ of the given LIP system (3.8) exists and is determined by (3.16), (3.17).

The proof is given [11].

3.2.2. Comparison with other methods. We compare the present method (referred to as method M3.3) with the iterative method of [8] (method M3.2) and the direct method of [17] (method M3.1) according to the following three criteria:

a) enclosure efficiency: tightness of the approximation of the solution set Σ obtained by the respective method;

b) computational efficiency;

c) applicability radius [8,11].

Comparison with the direct method of [17]

We first compare M3.3 with M3.1 which is also a direct method. It should, however, be stressed that method M3.1 yields an outer solution of (3.8) in the standard (nonparametric) form of an interval vector $\boldsymbol{x}$. We have the following result (which is a corollary of Theorem 3.2 and proved in [11]).

Corollary 3.3. The hull $\boldsymbol{x}(\boldsymbol{p})$ of the p-solution of $\boldsymbol{x}(p)$ of (3.8) obtained by method M3.3 is equal to the outer solution $\boldsymbol{x}$ of (3.8) obtained by method M3.1, i.e.

$$\boldsymbol{x}(\boldsymbol{p}) = \boldsymbol{x}. \tag{3.18}$$

It should be underlined that the p-solution $\boldsymbol{x}(p)$ is much more informative than the outer solution $\boldsymbol{x}$. Thus, knowledge of $\boldsymbol{x}(p)$ permits finding the corresponding intervals (3.4) which provide two-sided bounds on $\underline{x}_i^*$ and $\overline{x}_i^*$, respectively.

The advantage of method M3 over M1 reveals itself when solving problems other than finding an outer solution $\boldsymbol{x}$ of (3.8).To illustrate the above assertion, we consider the following parametric linear programming (PLP) problem

$$f(p) = c^T(p)x(p) \tag{3.19a}$$

where the constraint is the LIP system $A(p)x = b(p)$ with [11]

$$A(p)=\begin{bmatrix} p_1 & p_2+1 & -p_3 \\ p_2+1 & -3 & p_1 \\ 2-p_3 & 4p_2+1 & 1 \end{bmatrix}, \quad b(p)=\begin{bmatrix} 2p_1 \\ p_3-1 \\ -1 \end{bmatrix}, \quad p \in \boldsymbol{p}. \tag{3.19b}$$

The parameter interval vector $\boldsymbol{p}$ is given by its centre and radius

$$\breve{p}=(0.5\ \ 0.5\ \ 0.5), \quad \hat{p}=(0.5\ \ 0.5\ \ 0.5). \tag{3.19c}$$

Using $\breve{p}$ and $\hat{p}$, we transform (3.19b) into the equivalent form (as shown in [11]) to have $p_\mu \in [-1,1]$:

$$A(p)=\breve{A}+\sum_{\mu=1}^{m} A^{(\mu)} p_\mu, \quad a(p)=\breve{a}+A^0 p, \tag{3.20a}$$

$$\breve{A}=\begin{bmatrix} 0.5 & 1.5 & -0.5 \\ 1.5 & -3 & 0.5 \\ 1.5 & 3 & 1 \end{bmatrix}, \quad \breve{b}=(1\ \ -0.5\ \ -1)^T. \tag{3.20b}$$

The range of the solution of (3.19a) is the interval

$$\boldsymbol{f}^*\left(A(p),a(p),c(p),\boldsymbol{p}\right)=\left\{f=c^T(p)\,x: A(p)x=a(p),\ p\in\boldsymbol{p}\right\}$$

(denoted for shortness as $\boldsymbol{f}^*$). For simplicity, we have chosen

$$c^T=(1,1,1) \tag{3.20c}$$

(in the general case, $c=c(p)$ and $c(p)$ can be nonlinear functions). What we seek is to find an outer bound $\boldsymbol{f}$ on $\boldsymbol{f}^*$.

In the case of M1, $\boldsymbol{f}$ is given as

$$\boldsymbol{f}_1=\sum_i \boldsymbol{x}_i \tag{3.21}$$

where $\boldsymbol{x}_i$ are the components of the interval outer solution of (3.20). The bound $\boldsymbol{f}$ obtained by M3 and denoted $\boldsymbol{f}_3$ is found as the range of

$$f_3(p)=\sum_i \boldsymbol{x}_i(p), \quad p\in\boldsymbol{p}, \tag{3.22a}$$

that is

$$\boldsymbol{f}_3=f_3(\boldsymbol{p}). \tag{3.22b}$$

Now $\boldsymbol{f}_3$ is determined by (3.22a) using the components $\boldsymbol{x}_i(p)$ of the p-solution $\boldsymbol{x}(p)$.

To show quantitatively that (3.22b) is narrower than (3.21) we employ the merit figure

$$\eta_{31}\%=(1-r(\boldsymbol{f}_3)/r(\boldsymbol{f}_1)).100\%. \tag{3.23}$$

where $r(\boldsymbol{f}_1)$ and $r(\boldsymbol{f}_3)$ are the radii of $\boldsymbol{f}_1$ and $\boldsymbol{f}_3$, respectively. On account of Corollary 3.3

$$\boldsymbol{x}_i=\breve{x}_i+(\sum_j |L_{ij}|)[-1,1]+\hat{x}_i[-1,1],$$

hence, from (3.21)

$$\boldsymbol{f}_1 = f_0 + (\sum_{ij} \left| L_{ij}^0 \right|)[-1,1] + \hat{s}[-1,1], \tag{3.24a}$$

$$f_0 = \sum_i \breve{x}_i \ , \quad L_j^0 = \sum_i \left| L_{ij} \right|, \quad \hat{s} = \sum_i \hat{x} . \tag{3.24b}$$

Thus,

$$r(\boldsymbol{f}_1) = \sum_{ij} \left| L_{ij} \right| + \hat{s} . \tag{3.25}$$

To find $r(\boldsymbol{f}_3)$, we write $\boldsymbol{x}_i(p)$ as

$$\boldsymbol{x}_i(p) = \breve{x}_i + \sum_j L_{ij} p_j + \hat{x}_i \left[-1,1\right]$$

so

$$\boldsymbol{f}_3 = f_0 + \sum_j L_j p_j + \hat{s}\left[-1,1\right], \quad L_j = \sum_i L_{ij} .$$

Hence

$$r(\boldsymbol{f}_3) = \sum_j \left| L_j \right| + \hat{s} . \tag{3.26}$$

From (3.26) and (3.25)

$$r(\boldsymbol{f}_3) \le r(\boldsymbol{f}_1) \tag{3.27a}$$

since

$$\sum_j \left| \sum_i L_{ij} \right| \le \sum_{ij} \left| L_{ij} \right| . \tag{3.27b}$$

It is seen that method M3 has better enclosure efficiency than method M1.

We now recall the concept of the applicability radius r_a of a method M for solving a given problem P [8, 11]. Consider the family of boxes

$$\boldsymbol{p}(\rho) = \breve{p} + \rho\left[-\hat{p}, \hat{p}\right] \tag{3.28}$$

where $\breve{p}$ and $\hat{p}$ are given in (3.19c). We determine r_a of the respective method approximately by letting ρ increase with an increment $\Delta\rho$ until inapplicability of the method is reached.

Using (3.23), we now compute η_{31} for various ρ within the applicability radius r_a of method M3 which is [11]

$$r_a(M3) = 0.7449 . \tag{3.28a}$$

The corresponding values for η_{31} are given in the second row of Table 1.

Table 1. Enclosure efficiency ratio η_{31} of methods M3 and M1.

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.7448
η_{31} %	62.51	53.48	44.92	36.89	29.43	22.61	16.50	14.02

It is seen that η_{31} decreases as ρ grows. This is explained by the fact that the relative weight of the first term $\sum_j |L_j|$ in (3.26) decreases with respect to the second term $\hat{s}$ in function of ρ.

Comparison with the iterative method of [8]

The computation volume of the direct method is much smaller than that of the iterative method since the amount of computation needed in the former method is roughly the same as that required for each iteration of the latter method. Thus, the iterative method is bound to be more expensive than the direct method and this discrepancy will become more pronounced as the radius of $\boldsymbol{p}$ approaches the applicability radius r_a.

Let $\boldsymbol{x}^{(2)}(p)$ and $\boldsymbol{x}^{(3)}(p)$ denote the *p*-solutions of (3.20) obtained by methods M3.2 and M3.3, respectively. The enclosure efficiency of the two *p*-solutions will be compared solving the PLP problem (3.20). In this case, we compare the outer solutions $\boldsymbol{f}_3$ with $\boldsymbol{f}_2$ determined in a similar way by

$$\boldsymbol{f}_2(p) = \sum_i \boldsymbol{x}_i^{(2)}(p), \quad p \in \boldsymbol{p}, \tag{3.29a}$$

$$\boldsymbol{f}_2 = \boldsymbol{f}_2(\boldsymbol{p}) . \tag{3.29b}$$

To quantitatively assess the superiority of M2 over M3, we use the merit figure

$$\eta_{23}\% = (1 - r(\boldsymbol{f}_2) / r(\boldsymbol{f}_3)).100\% . \tag{3.30}$$

To show the dependence of η_{23} on the parameter width ρ, we need the applicability radius $r_a(\mathrm{M2})$ of method M2. The numerical experiment has shown that for method M2 (approximately)

$$r_a(\mathrm{M2}) = 0.71 \tag{3.31}$$

(the iterative process becomes divergent for $\rho = 0.72$). It is seen from (3.28a) and (3.31) that the present direct method M3 has a slightly larger applicability radius than the iterative method M2.

We now show the dependence of η_{23} on ρ from $\rho = 0.1$ up to $\rho = 0.7$. The corresponding values of η_{23} are given in the second row of Table 2. As expected, the iterative method M2 provides tighter outer bound $\boldsymbol{f}^{(2)}(p)$ as compared to $\boldsymbol{f}^{(3)}$.

Table 2. Comparison of the computational characteristics of methods M2 and M3.

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7
η_{23} %	13.55	21.33	26.21	29.33	31.21	32.14	32.18
τ	1.79	1.87	2.82	2.86	3.38	5.39	16.49

The two methods are also assessed as regards their computational efficiency using the index $\tau = t_2 / t_3$ where t_2 and t_3 denote the respective computer time taken by M2 or M3. The related data are listed in the third row of the table. It should be kept in mind that $t_3 = 0.0050$ s and remains constant for all ρ.

The data in Table 2 show that the better enclosure efficiency of method M2 is obtained at the cost of much larger computer times, especially for ρ close to $r_a(\mathrm{M2})$. Also $r_a(\mathrm{M2}) < r_a(\mathrm{M3})$. Therefore, in some cases, it may be preferable to use M3 rather than M2.

3.3. A new method

3.3.1. Computational scheme. In this section, we present a method for determining a *p*-solution to the general form NLIP system

$$f(x,y)=0, \quad y\in \boldsymbol{y} \tag{3.32}$$

which is based on the following ideas.

1). First we observe that (3.32) defines x as an implicit function $x = g(y)$ with $g:\boldsymbol{y}\subset R^m \to R^n$. It is seen that the solution set Σ of (3.32), in fact, is the image of y under g. Thus, the problem of determining a *p*-solution to (3.32) consists in finding a LIP enclosure of $g(y)$ over $\boldsymbol{y}$, i.e.

$$\boldsymbol{x}(y)=\tilde{\boldsymbol{g}}(y)=Ly+\boldsymbol{l}, \quad y\in \boldsymbol{y} . \tag{3.33}$$

However, since g(y) is not known, we suggest the following approach to circumventing this fact.

2). We first make the following

Assumption 3.1. An outer interval approximation $\boldsymbol{x}^0$ of the solution set Σ of (3.32) is available.

3). The variables $\boldsymbol{x}^0$ and $\boldsymbol{y}$ in (3.32) are then treated as independent, forming an augmented size variable vector $t = (x, y)$. From this point of view, (3.32) is replaced by

$$f(x,y)=0, \quad x\in \boldsymbol{x}^0, \quad y\in \boldsymbol{y}. \tag{3.34}$$

4). The variables $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$ are written in normalized form

$$x_i=\check{x}_i+\hat{x}_i q_i, \quad q_i\in \boldsymbol{q}_i=[-1,1], \tag{3.35a}$$

$$y_i=\check{y}_i+\hat{y}_i p_i, \quad p_i\in \boldsymbol{p}_i=[-1,1]. \tag{3.35b}$$

It is seen that each x_i, y_i is in affine form. Now each $f_i(t), t\in \boldsymbol{t}$ in (3.32) is modified to

$$f_i(q,p)=0, \quad q\in[-1,1]^n, \quad p\in[-1,1]^m . \tag{3.36a}$$

Since $f_i(q,p)$ is a function in explicit form, it can be approximated by the LIP enclosure

$$\boldsymbol{l}_i(q,p)=l_i+A_i^q q+A_i^p p+\boldsymbol{l}_i \tag{3.36b}$$

where A_i^p and A_i^q are rows of size *n* and *m*, respectively. Thus, we can enclose the modified system

$$f(q,p)=0 \tag{3.37a}$$

by

$$\boldsymbol{l}(q,p)=\check{l}+A^q q+A^p p+\hat{l}[-1,1]=0, \quad q\in \boldsymbol{q}, \quad p\in \boldsymbol{p} \tag{3.37b}$$

where A^q and A^p are matrices of size $n\times n$ and $n\times m$, respectively.

5). We need the following

Assumption 3.2. The matrix A^q is nonsingular.

Hence from (3.37b) we now determine

$$q=\check{q}+Lp+\hat{q}[-1,1], \quad p\in \boldsymbol{p}, \tag{3.38}$$

$$\check{q}=-(A^q)^{-1}\check{l}, \quad L=-(A^q)^{-1}A^p, \quad \hat{q}=\left|(A^q)^{-1}\right|\hat{l} .$$

It is seen that q is a member of the affine form

$$\langle q\rangle=\check{q}'+L'p+\hat{q}'[-1,1], \quad p\in \boldsymbol{p}. \tag{3.39}$$

6). Using (3.35) we compute

$$x(p) = \breve{x} + Lp + \hat{x}[-1,1]\ ,\ p \in \boldsymbol{p} \tag{3.40}$$

with components

$$x_i = \breve{x}_i + L_i p + \hat{x}_i[-1,1],\quad \breve{x}_i = \breve{x}_i^0 + \hat{x}_i^0 \breve{q}_i',\quad L_i = \hat{x}_i^0 L_i',\quad \hat{x}_i = \hat{x}_i^0 \hat{L}_i' . \tag{3.40a}$$

It is seen that (3.40) is a LIP form.

7). Let $\boldsymbol{x}$ be the interval hull of (3.40). We make the following

Assumption 3.3.

$$\boldsymbol{x} \subseteq \boldsymbol{x}^0 . \tag{3.41}$$

We have the following

Theorem 3.4. If Assumptions 3.1 to 3.3 hold, then $\boldsymbol{x}(p)$ obtained by (3.35) to (3.40) is a parameterized solution of (3.32).

If $\boldsymbol{x}$ is narrower than $\boldsymbol{x}^0$ and the reduction is larger than a threshold ε_r, then $\boldsymbol{x}$ can be renamed $\boldsymbol{x}^0$ and a new iteration can be resumed from (3.35). This continues until some stopping criterion is met. We have chosen a simple stopping criterion: the iterations are terminated as soon as the relative distance between two successive iterates satisfies the condition

$$d_r(\boldsymbol{x}^0, \boldsymbol{x}) \le \varepsilon_1 , \tag{3.42}$$

$$d_r(\boldsymbol{a},\boldsymbol{b}) = \max\left\{\max_i |\underline{a}_i - \underline{b}_i|,\ \max_i |\overline{a}_i - \overline{b}_i|\right\} / \max(rad(b_i)) .$$

Thus, we have the following algorithm A1 of the present method referred to as Method M1.

Algorithm A1. To simplify the presentation of the algorithm, we assume the reduction is larger than ε_r.

Step 0. Compute an outer solution $\boldsymbol{x}^0$ of system 3.32 using a known interval method [1].

Step 1. Determine the LIP solution $\boldsymbol{x}(p)$ and its hull $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p})$ using formulae (3.35) to (3.40).

Step 2. If (3.42) is satisfied, go to Termination; otherwise let $\boldsymbol{x}^0 := \boldsymbol{x}$ and go to Step 1.

Termination: a LIP solution $\boldsymbol{x}(p)$ of the form (3.40) has been determined.

3.3.2. Numerical example. We illustrate method M1 using a simple nonlinear system having n = 2 and m = 2. The algorithm of the method was programmed in MATLAB environment using the toolbox IntLab.8 [7] to carry out the interval calculations involved. The AA arithmetic was implemented by the *affari* toolbox [2]. The program was run on a 1.7 GHz PC computer.

We consider the nonlinear parametric system

$$f_1(x, y) = -y_1 x_1 + x_2^2 = 0 , \tag{3.43a}$$

$$f_2(x, y) = -y_2 x_1 + 0.5 y_2 x_1^2 + x_2^2 - y_2 = 0 \tag{3.43b}$$

where the parameters vary within the intervals

$$y_1 \in \boldsymbol{y}_1 = [15.8, 16.2],\quad y_2 \in \boldsymbol{y}_2 = [18, 19.6] \tag{3.44}$$

System (3.43) has been used as a portion of a bigger over-constrained system in [18].

We first determine a p-solution related to system (3.43), (3.44) using method M1.

Determining a p-solution

We begin by determining the p-solution corresponding to the first iteration of algorithm A1 ($k = 1$).

It is known [18] that the box

$$\boldsymbol{x}_1^0 = [1.2, 1.7], \quad \boldsymbol{x}_2^0 = [-5.8, -4.7] \tag{3.45}$$

contains the solution set Σ of (3.43), (3.44). Thus, choosing $\boldsymbol{x}^0$ from (3.45) to be used in algorithm A1, we satisfy Assumption 1. Next, we are to compute the linear form (3.37b) related to system (3.43) to (3.45). To this end, we apply *affari* toolbox to the above system. The linear enclosure $\boldsymbol{l}_1(q,p)$ of the function $f_1(x,y)$ is computed using AA for the product y_1x_1 and Chebyshev approximation for the nonlinear univariate function $f_{12}(x_2) = x_2^2$. There are two ways to compute $\boldsymbol{l}_2(q,p)$:

apply AA to the sub-expression $y_2(x_1(0.5x_1 - 1) - 1)$;

find the Chebishev approximation for the nonlinear univeriate function $f_{21}(x_1) = x_1(0.5x_1 - 1) - 1$ and compute the product $y_2 f_{21}(x_1)$ using *affari*.

Numerical evidence has shown that better results (narrower intervals by approximately 11 %) are obtained if the latter approach is used. Thus, the system we consider is

$$f_1(x,y) = -y_1x_1 + x_2^2 = 0, \tag{3.46a}$$

$$f_2(x,y) = y_2 f_{21}(x_1) + x_2^2 = 0, \quad f_{21}(x_1) = x_1(0.5x_1 - 1) - 1, \tag{3.46b}$$

$$y \in \boldsymbol{y}, \quad x \in \boldsymbol{x}^0 \tag{3.46c}$$

where $\boldsymbol{y}$ and $\boldsymbol{x}^0$ are given in (3.44) and (3.45), respectively. The related system (3.37b) is

$$\breve{l} = \begin{pmatrix} -4.5137 \\ -1.7110 \end{pmatrix}, \quad A^q = \begin{pmatrix} -4.000 & -5.775 \\ 2.115 & -5.775 \end{pmatrix}, \quad A^p = \begin{pmatrix} 0.2900 & 0 \\ 0 & 1.1065 \end{pmatrix}, \quad \hat{l} = \begin{pmatrix} 0.20125 \\ 0.54750 \end{pmatrix}.$$

Since matrix A^q is non-singular, the second Assumption 2 is also satisfied. Hence, we can compute

$$\boldsymbol{x}(p) = \breve{x} + Lp + \hat{x}[-1,1], \quad p \in \boldsymbol{p} \tag{3.47}$$

where

$$\breve{x} = \begin{pmatrix} 1.5645850 \\ -4.9947248 \end{pmatrix}, \quad L = \begin{pmatrix} -0.0118561 & 0.0523712 \\ -0.0095526 & -0.0689328 \end{pmatrix}, \quad \hat{x} = \begin{pmatrix} 0.0877044 \\ 0.1192227 \end{pmatrix}. \tag{3.47a}$$

Finally

$$\boldsymbol{x} = \begin{pmatrix} [1.4768806, \ 1.6243233] \\ [-5.1139475, -4.8755021] \end{pmatrix} \tag{3.48}$$

and it is seen that $\boldsymbol{x} \subset \boldsymbol{x}^0$ so Assumption 3 is also valid. Thus, the LIP form (3.47) determines the *p*-solution sought.

The subsequent iterations of algorithm A1 yield better enclosures. The improvement for two successive iterations is assessed by

a) the relative reduction in the radii $r_i^0 = rad(\boldsymbol{x}_i^0)$ with the respect to the radii $r_i = rad(\boldsymbol{x}_i)$ using by the following figure of merit $\varphi_i = (r_i^0 - r_i)/r_i^0 = 1 - r_i / r_i^0$ %;

b) the relative reduction in the respective volumes $V^0 = 4rad(\boldsymbol{x}_1^0)rad(\boldsymbol{x}_2^0)$ and $V = 4rad(\boldsymbol{x}_1)rad(\boldsymbol{x}_2)$ given by $\theta = (V^0 - V)/V^0 = 1 - r_1r_2 / r_1^0r_2^0$ %;

The values of φ_i and θ decrease as k grows. We first report data for $k = 1$:

$$\varphi_1 = 64.9\,\%\,,\quad \varphi_2 = 78.3\,\%\,,\quad \theta = 92.4\,\%\,.$$

It is seen that already at the first iteration, the starting box (3.44) has been contracted considerably to a narrow box (3.48).

Next for $k = 4$

$$\varphi_1 = 0.34\,\%\,,\qquad \varphi_2 = 0.37\,\%\,,\quad \theta = 0.92\,\%$$

for the relative reduction between the two third and fourth iterations. Since φ_1, φ_2 and θ are lesser than 1 per cent, the iterations have been terminated and the data for k = 4 have been accepted as the final p-solution with

$$\breve{x} = \begin{pmatrix} 1.5704569 \\ -5.0123972 \end{pmatrix},\quad L = \begin{pmatrix} -0.0117529 & 0.0400120 \\ -0.0125726 & -0.0638622 \end{pmatrix},\quad \hat{x} = \begin{pmatrix} 0.0538664 \\ 0.0795133 \end{pmatrix}. \tag{3.49}$$

Hence

$$\boldsymbol{x} = \begin{pmatrix} [1.5704569,\ 1.6243233] \\ [-5.0919105, -4.9328839] \end{pmatrix}. \tag{3.50}$$

It is seen that method M1 has a good convergence: the box (3.48) (related to the p-solution at the first iteration) has been contracted to a much smaller box (3.50) (related to the p-solution at the fourth iteration). Indeed, the corresponding data for $\varphi_i = (r_i^{(1)} - r_i^{(4)}) / r_i^{(1)}$ and $\theta = (V^{(1)} - V^{(4)}) / V^{(1)}$ are

$$\varphi_1 = 69.3\,\%\,,\quad \varphi_2 = 33.3\,\%,\quad \theta = 79.5\,\%.$$

Properties

We first illustrate the validity of Theorem 3.4, considering the interval $\boldsymbol{e}_2^l = [\underline{e}_2^l, \overline{e}_2^l]$ only. The lower end $\underline{e}_2^l$ is given by the lower end $\underline{x}_2$ of the second interval in (3.50) while $\overline{e}_2^l = \underline{x}_2 + 2\hat{x}_2$ hence

$$\boldsymbol{e}_2^l = [-5.1139, -4.8755].$$

The lower end $\underline{x}_2^*$ of the IH solution $\boldsymbol{x}_2^*$ is approximately

$\underline{x}_2^* = -5.0884$

and it is seen that indeed

$\underline{x}_2^* \in \boldsymbol{e}_2^1$.(3.51)

Next, we show the validity of (3.7b) using data in (3.44) and (3.45) (related to the first iteration of method M1). We have obtained

$$\boldsymbol{f}_1(\boldsymbol{p}) = [-1.063336, 0.905381],\ \ \boldsymbol{f}_1(\boldsymbol{x},\boldsymbol{y}) = [\,-5.45,\ 14.68]\,,\ \ \langle \boldsymbol{f}_1 \rangle = [-5.7525, 14.78],$$

$$\boldsymbol{f}_2(\boldsymbol{p}) = [-4.486993, 2.390259],\ \ \boldsymbol{f}_2(\boldsymbol{x},\boldsymbol{y}) = [-10.838, 12.4],\ \langle \boldsymbol{f}_2 \rangle = [-7.5305, 11.05]\,.$$

It is seen that indeed $\boldsymbol{f}_1(\boldsymbol{p})$ and $\boldsymbol{f}_2(\boldsymbol{p})$ provide the narrowest inclusion of the functions $f_i(x,y)$, $x \in \boldsymbol{x}$, $y \in \boldsymbol{y}$, i=1, 2 and the improvement is considerable.

Remark 3.1. The p-solutions considered so far are defined as a linear interval parametric (LIP) form. A p-solution in a corresponding quadratic interval (QIP) form has recently been suggested in [9]. The properties of the QIP solution are similar to those of the LIP solution.

4. Applications

In this section, we present a general approach to formulating and solving a class of global optimization problems: find the global minimum of the scalar function $f_0 : R^n \times R^m \to R$

$$f_0^* = \min f_0(x, p) \tag{4.1a}$$

subject to the constraint

$$f(x, p) = 0, \quad p \in \boldsymbol{p}. \tag{4.1b}$$

Each individual problem is set up by specifying the functions $f_0(x, p)$ and $f(x, p)$. This approach has already been employed in [19] in the conventional parametric setting. In this paper, it is further developed as follows:

(i) use of a p-solution of (4.1b);

(ii) use of a constraint-satisfaction technique.

Remark 4.1. If the constraint-satisfaction technique is not (sufficiently) efficient, modified monotoncity conditions [20] could be employed. These should, however, be implemented using the p-solution, which is expected to improve their efficiency.

Remark 4.2. So far the concept of p-solution $\boldsymbol{x}(p)$ has been introduced as an outer LIP enclosure of the solution set Σ of the given LIP system (4.1b). It will be now extended to problem (4.1). To find an enclosure $\boldsymbol{f}_0(p)$ of f_0^* in a LIP form, we substitute x in (4.1a) for $\boldsymbol{x}(p)$ and carry out the AA operations involved. Thus, we obtain $\boldsymbol{f}_0(p)$ that is of the form required:

$$\boldsymbol{f}_0(p) = \breve{f}_0 + L_o p + \hat{f}_0[-1,1], \quad p \in \boldsymbol{p}. \tag{4.1c}$$

We will refer to $\boldsymbol{f}_0(p)$ as a (linear) p-solution of the global optimization problem (4.1a), (4.1b). This solution has enclosing properties as regards f_0^* which are similar to those of $\boldsymbol{x}(p)$ with respect to Σ. Thus, the corresponding interval

$$\boldsymbol{e}_0^{(1)} = [\underline{f}_0, \ \underline{f}_0 + 2\hat{f}_0] \tag{4.1d}$$

is bound to contain f_0^*. That is why $\boldsymbol{e}_0^{(1)}$ can be used to construct a constraint equation.

The above general approach will be illustrated in the next subsections.

4.1. Tolerance analysis problems for electric circuits

The LIP models have been introduced only recently in electric circuit theory [21] for addressing the problem of worst-case tolerance analysis (WCTA). In this framework it is preferable to use the nodal analysis (NA) as well as modified nodal analysis (MNA) equations since these lead to LIP systems of the smallest size possible.

It should be stressed that so far the LPD-LIP models have been applicable solely to linear circuits.

Known interval methods provide all three types of solutions: outer solutions, exact (interval hull) solutions and inner estimation solutions. Most of the publications suggest outer solutions. The existing methods for exact solutions are based on the use of respective global or modified monotonicity conditions which verification is, however, computationally rather cumbersome.

4.1.1. Direct current (DC) circuits. In view of the method suggested in Section 3.3, the WCTA problem can be formulated in a rather general manner using the NLIP description (4.1b). Thus, WCTA can be carried out for nonlinear circuits also. However, for simplicity we present the linear parametric dependence (LPD) description

$$A(p)x = b(p), \quad p_\mu \in \boldsymbol{p}_\mu, \quad \mu = 1,...,m \tag{4.2}$$

$$a_{ij}(p) = \sum_{\mu=1}^{m} a_{ij\mu} p_\mu, \quad b_i(p) = \sum_{\mu=1}^{m} \beta_{i\mu} p_\mu. \tag{4.2a}$$

The nodal analysis (NA) as well as modified nodal analysis (MNA) equations can now be used. In the case of DC circuits described by MNA equations, the LIP system (4.2) has a specific structure

$$A(p) = G(p), \quad b(p) = J(p) \tag{4.2b}$$

where $G_{ii}(p)$ and $G_{ij}(p)$ are the proper and mutual conductances while $J_i(p)$ is the corresponding equivalent current source; x is the vector of the node voltages. The LIP matrix $G(p)$ is symmetric.

As is well known, the exact solution of the WCTA problem is to determine the hull solution $\boldsymbol{x}^*$ related to (4.2b). According to [20] each component $\boldsymbol{x}_k^* = [\underline{x}_k^*, \overline{x}_k^*]$ is computed separately. Each end of $\boldsymbol{x}_k^*$, in turn, can be found by solving the following two parametric linear programming (PLP) problems [20], [21]:

$$x_{k-}^* = \min e_k^T x : \quad G(p)x = J(p), \quad p \in \boldsymbol{p}, \tag{4.3a}$$

$$x_{k+}^* = -\{\min(-e_k^T)x : G(p)x = J(p), p \in \boldsymbol{p}\} \tag{4.3b}$$

(e_k is the kth column of the identity matrix). It is seen that in the framework of the general approach, the function $f_0(x, p)$ is a linear function $f_0(x) = c^T x$ with $c^T = e^T$.

Consider the PLP problem (4.3a). According to the general approach, we first determine a p-solution of system (4.2), (4.2b)

$$\boldsymbol{x}_k(p) = \breve{x}_k + \sum_{j=1}^{m} l_{kj} p_j + \hat{x}_k[-1, 1], \ p \in \boldsymbol{p}. \tag{4.4}$$

Next, we have to construct the constraint equation. The interval $\boldsymbol{e}_k^{(1)} = [\underline{x}_k, \underline{x}_k + 2\hat{x}_k]$ is used to this end

$$\breve{x}_k + \sum_{j=1}^{m} l_{kj} p_j + \hat{x}_k[-1, 1] = \boldsymbol{e}_k^{(1)}, \ p \in \boldsymbol{p}. \tag{4.5}$$

Now a simple constraint satisfaction technique (given in Appendix B of [8]) is applied, trying to reduce the current domain $\boldsymbol{p}$ to a narrow domain $\boldsymbol{p}'$.

Based on the theoretical results and judging from some preliminary results (solving the LPL problems from [22]), the new method is bound to outperform method M3 from [21] as regards the scope of its applicability.

4.1.2. Alternating current (AC) circuits. In this case, the MNA system is in complex form

$$YV = J. \tag{4.6}$$

where Y is a complex $n \times n$ matrix and J is a complex vector depending both on respective real G_k or complex jB_k conductances; V is a complex vector whose components are $\dot{V}_k = V_k^a + jV_k^r$. The complex system (4.6) is now replaced by the following real system of double size $2n \times 2n$

$$A(p)x = b(p), \quad p \in \boldsymbol{p} \tag{4.7a}$$

where

$$A(p) = \begin{bmatrix} G(p) & -B(p) \\ B(p) & G(p) \end{bmatrix}, \quad x = \begin{bmatrix} V^a \\ V^r \end{bmatrix}, \quad b = \begin{bmatrix} J^a(p) \\ J^r(p) \end{bmatrix}. \tag{4.7b}$$

As an illustration we consider the following

Example 4.1. This example refers to the AC (a notch filter) circuit considered in [21]. It has $n' = 4$

nodes and $m = 8$ branches. The nominal (centre) values of the interval element parameters R_μ and C_μ are given in [21]. The WCTA problem consists in finding the exact solutions related to the real V_a and imaginary V_r part of the output voltage V_3 for several values of the tolerances on R_μ and C_μ. The complex system obtained by nodal analysis involves $n' = 4$ equations. The corresponding real system has $n = 2n' = 8$ and is

$$\begin{bmatrix} G(p) & -B(p) \\ B(p) & G(p) \end{bmatrix} x = J, \quad p \in \boldsymbol{p}, \tag{4.8a}$$

$$G(p) = \begin{bmatrix} p_1 + p_4 & -p_4 & 0 & 0 \\ -p_4 & p_4 + p_5 & -p_5 & 0 \\ 0 & -p_5 & p_3 + p_5 & 0 \\ 0 & 0 & 0 & p_2 \end{bmatrix}, \quad B(p) = \begin{bmatrix} p_7 & 0 & 0 & -p_7 \\ 0 & p_6 & 0 & 0 \\ 0 & 0 & p_8 & -p_8 \\ -p_7 & 0 & -p_8 & p_7 + p_8 \end{bmatrix}. \tag{4.8b}$$

The tolerance radius $\hat{p}_\mu$ of each p_μ is defined as a certain percentage of $\breve{p}_\mu$, i.e.

$$\hat{p}_\mu = t\breve{p}_\mu, \quad \mu = 1, \ldots, m. \tag{4.8c}$$

The output variables V_a and V_r are given by the components x_3 and x_7 of the solution vector x, respectively, hence the tolerances sought on V_a and V_r are given by the hull solutions $\boldsymbol{x}_3^*$ and $\boldsymbol{x}_7^*$.

We give results on $\boldsymbol{V}_a^*$ obtained by method M3 of [21] which employs local monotonicity conditions. It is capable of finding $\boldsymbol{V}_a^*$ (i.e., the hull solution $\boldsymbol{x}_3^* = \left[\underline{x}_3^*, \overline{x}_3^*\right]$ only for t up to $t = 0.05$ (using an increment $\Delta t = 0.01$). Thus, for $t = 0.07$ the method can produce only an interval which contains the exact solution end $\overline{x}_3^*$. Method M3 is inapplicable for t = 0.1 since none of the monotonicity conditions used is satisfied.

It would be interesting to apply the p-solution approach to this example and compare performance. We expect better results for the new approach.

4.2. Power consumption analysis

In [23] the following problem was considered: given a linear DC circuit whose parameters p (resistors and sources) vary within $\boldsymbol{p}$, determine the range $\boldsymbol{P}^*$ of the electric power $P(p)$, $p \in \boldsymbol{p}$ consumed in the circuit. This WCTA problem differs from those presented in Sect. 4.1 in that the output characteristic $f_0(x, p)$ is a nonlinear function of the variables I_k involved

$$P(p) = \sum_{k=1}^{m} p_k I_k^2(p), \ p \in \boldsymbol{p}. \tag{4.9}$$

It is shown in [23] that the power range sought can be computed as the range of an associated interval linear programming (ILP) problem. The transition to the latter ILP problem is only possible if the circuit has one of the following two mathematical models:

A. The tableau description. In this case the parameters are independent but their number m' is very large - $m' = m + n$. The size of the system (denoted S1) is equal to m'.

B. The hybrid equation description (system S2). Once again the parameters are independent but still their number is rather large - $m' = m$ and the size of S2 is m.

It should be stressed that the most economical description S3 using nodal analysis (which has only n variables and typically $n \ll m$) is not admissible in the present approach.

The case of AC circuits has been considered in [24].

Here we suggest a new method for solving the power range problem which is applicable for both DC and AC circuits. For simplicity it will be presented only for the DC case.

The method has the following computational scheme.

1). Find the *p*-solution $V(p)$, $p \in \boldsymbol{p}$ of the nodal equation system (4.2), (4.2b) where $x = V$.

2). Let m′ denote the number of branches having voltage sources E_j, $j \in j_E$ where j_E is the set of indices related to such branches. Typically, $j_E \ll m$. Denote the current I_j in the jth branch flowing from node *v* to node μ as $I_{v\mu}$. Next, find its *p*-expression

$$I_j(p) = I_{v\mu}(p) = G_{v\mu}(E_{v\mu} + V_{v\mu}(p)). \tag{4.10}$$

3). Find the *p*-form of *P*(*p*) as

$$P(p) = \sum_{j \in jE} E_j I_j(p) \tag{4.11}$$

4) Determine the range $P^* = \left[\underline{P}^*, \overline{P}^* \right]$ by applying the constraint-satisfaction technique separately to determine $\underline{P}^*$ and $\overline{P}^*$, respectively.

In contrast to [23,24], the present general method is applicable for solving power range determination problems described by the much smaller description system S3. Also, its numerical complexity is only polynomial, thus making possible the treatment of larger size circuits.

4.3. Truss analysis

Truss analysis is a well-established research aria in civil and mechanical engineering (see, e.g., [25-27]). Accounting for the uncertain parameters leads to LIP systems. Here we focus on [27] where an interesting example of a NLP system is considered.

Example 4.1 (Example 3.2 in [27]). A two-bay two-story frame is studied. The NLP system modeling the frame consists of 18 equations whose coefficients are rational functions of 13 uncertain parameters. The results refer to the displacements and rotations of selected nodes.

Initially, all parameters are taken to have 1% tolerance intervals. The results are obtained using a parametric fixed-point-iteration solver presented there. Guaranteed outer enclosure [u] of the system response and a corresponding inner estimation [v] of the solution set hull are presented. Next, the system is solved with the same 1% tolerance for the element material properties but with increased 10% tolerance intervals for the spring stiffness and all applied loadings. Results are reported for nodes 1 and 3. The interval bounds for the system response are esteemed “reasonable but not quite sharp”. To get better solution enclosures, monotonicity conditions will respect to the loading parameters $w_1,..,w_4, f_1, f_2$ have been proved. Then quite sharp results are obtained solving corresponding parametric systems involving a reduced number of parameters.

Remark 4.3. The reported percentages of 1% and 10% tolerances are related to the width of the uncertain parameter intervals. However, in engineering the accepted manner to define a tolerance is to specify it as a given percentage of the interval radius. In that sense, the tolerances used above should be downgraded to 0.5% and 5%, respectively.

The above example would be an excellent bench test for comparing the computational efficiency of the standard approach and the new strategy using *p*-solutions.

4.4. Eigenvalue range determination

Eigenvalue range determination (estimation) of interval or parametric matrices has numerious applications: assessing the stability of dynamic systems or analysis of the natural frequencies in mechanical structures are just two characteristic examples.

For simplicity, here we consider the problem of bounding or determining the ranges of real eigenvalues in LPD systems, i.e. the standard eigenvalue problem

$$A(p)x = \lambda x, \quad p \in \boldsymbol{p}, \tag{4.12}$$

$$A(p) = A^{(0)} + \sum_{\mu=1}^{m} A^{(\mu)} p_{\mu}, \qquad p_{\mu} \in \boldsymbol{p}_{\mu} \tag{4.12a}$$

where $A^{(0)}$, $A^{(\mu)}$ are real $n \times n$ matrices. Without loss of generality, the intervals $\boldsymbol{p}_{\mu}$ are assumed symmetric and of unit radius. We consider how to determine the lower endpoint $\underline{\lambda}_k^*$ and upper endpoint $\overline{\lambda}_k^*$ of the range $\boldsymbol{\lambda}_k^* = [\underline{\lambda}_k^*, \overline{\lambda}_k^*]$ of an eigenvalue λ_k.

Let $x(p)$ and $y(p)$ denote the unique (after normalization) right eigenvector and left eigenvector associated with $\lambda_k(A(p))$ for a fixed $p \in \boldsymbol{p}$, i.e.

$$A(p)x = \lambda x, \quad p \in \boldsymbol{p}, \tag{4.13a}$$

$$A^T(p)x = \lambda x, \quad p \in \boldsymbol{p}. \tag{4.13b}$$

Also, let $\boldsymbol{x}^*$, $\boldsymbol{y}^*$ and $\boldsymbol{x}$, $\boldsymbol{y}$ be the corresponding ranges and outer bounds. The following sign-constancy assumption is necessary.

Assumption 4.1 (Assumption 4 in [28]). It is assumed that

$$0 \notin \boldsymbol{z}_{\mu} = \boldsymbol{y}^T A^{(\mu)} \boldsymbol{x}, \qquad \mu = 1, \ldots, m. \tag{4.14}$$

The following theorem is proved in [28].

Theorem 4.1 (Theorem 3 in [28]). If conditions (4.14 hold, then

$$\underline{\lambda}_k^* = \lambda_k(A(p^{'})) \tag{4.15a}$$

where $p^{'}$ has components

$$p_{\mu}^{'} = \begin{cases} \underline{p}_{\mu}, & \text{if } \boldsymbol{z}_{\mu} > 0 \\ \overline{p}_{\mu}, & \text{if } \boldsymbol{z}_{\mu} < 0 \end{cases} \tag{4.15b}$$

while

$$\overline{\lambda}_k^* = \lambda_k(A(p")) \tag{4.16a}$$

where $p"$ has components

$$p_{\mu}^{"} = \begin{cases} \overline{\lambda}_{\mu}, & \text{if } \boldsymbol{z}_{\mu} > 0 \\ \underline{\lambda}_{\mu}, & \text{if } \boldsymbol{z}_{\mu} < 0 \end{cases}. \tag{4.16b}$$

The extension of the present method to a new method using *p*-solutions $\boldsymbol{x}(p)$ and $\boldsymbol{y}(p)$ rather than interval vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is straightforward. Indeed, $\boldsymbol{x}(p)$ and $\boldsymbol{y}(p)$ can be determined from corresponding LIP systems as shown in [28], [29]. Then the new sign-constancy condition will be as follows.

Assumption 4.2. Let

$$\boldsymbol{z}_{\mu}(p) = \boldsymbol{y}^T(p) A^{(\mu)} \boldsymbol{x}(p). \tag{4.17a}$$

It is assumed that

$$0 \notin z_\mu(p), \quad \mu = 1, \ldots, m\,. \tag{4.17b}$$

The simplest (but not necessary the best) way to implement (4.17b) is to replace $z_\mu(p)$ with $z_\mu(\boldsymbol{p})$.

An improved version of the above problem has also been developed for the case where some of the sign-constancy conditions are violated; determining the ranges of the real and imaginary parts of complex eigenvalues has also been considered [28]. These modifications can be easily implemented using the p-solution approach.

Finally, we mention the paper [30] where the concept of regularity radius $r^*(A(\boldsymbol{p}))$ of LPD matrices $A(\boldsymbol{p})$ is introduced. It is shown that the numerical value of $r^*(A(\boldsymbol{p}))$ can be determined as the real maximum magnitude eigenvalue of a specific interval parametric generalized eigenvalue problem. The relationship between $r^*(A(\boldsymbol{p}))$ and certain real eigenvalue range problems is also discussed. See also [31].

It should also be noted that an alternative regularity measure, so-called regularity margin, of the same class of LIP matrices has been recently introduced in [32].

5. Conclusions

It has been shown that the *p*-solution recently introduced for the case of linear interval parametric (LIP) systems having linear parametric dependences (LPD) or nonlinear parametric dependences (NLPD) can also be defined and determined for the general class of nonlinear interval parametric (NLIP) system. This is done using appropriate transformations of the original NLIP system into an associated LIP system (Sect.2). The enclosing properties of the *p*-solutions have been considered in Section 3.1. Of the known methods for determining *p*-solutions that are only applicable to LIP systems, the simplest direct method is presented in Section 3.2. A new method for computing p-solutions in the case of NLIP systems has been suggested in Section 3.3. It should be underlined that the computational complexity of all methods is polynomial in the size of the parametric system studied.

Several applications in the fields of electrical, civil and mechanical engineering such as worst-case tolerance analysis of electric circuits, determining the power consumption range in direct or alternating current circuits, truss analysis of mechanical structures, eigenvalue range determination for parametric matrices have been given in Section 4. It has been shown that the new methods employing p-solutions have improved computational efficiency and, hence, larger radii of applicability as compared to other known methods.

It is hoped that the present survey will help promote the use of the new *p*-solution approach for solving various engineering problems involving uncertainties and risks.

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