Worst-case solution spaces for systems design under uncertainties

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Abstract

A general goal in systems design is to find a design that guarantees an optimal performance of the system. In practice, this cannot be reached as uncertainties occur which are responsible for a crucial reduction of the performance. Therefore, uncertainties must be taken into account in order to reach optimal or nearly optimal performance. In robust design optimization this is often addressed by considering uncertainties in uncontrollable parameters and controllable design variables that serve both as an input for the system performance. However, a mathematical quantification of the uncertainties is required which is in general unavailable at an early design phase of development due to a lack-of-knowledge situation. In this work, these uncertainties are regarded by assigning intervals to the uncontrollable parameters and a maximal worst-case solution space, also in the form of intervals, to the design variables. This worst-case solution space contains only designs which satisfy a performance requirement for all possible uncontrollable parameters. Furthermore, the center of the solution space is the design which tolerates the largest uncertainties in its variables.

Keywords: Systems design, interval uncertainties, lack of knowledge, solution spaces, optimization

1 Introduction

The method of using maximal box-shaped solution spaces for systems design was introduced in [9]. It is a set-based design approach (see [8] for an introduction) which provides a set of designs that fulfill all crucial requirements on the systems responses. Thus, the decision for a particular design can be postponed. As permissible intervals for design variables are made available with this approach, the decision for a design can also be decoupled between the single design variables.
Furthermore, variations in design variables can be handled with these solution spaces. Beneath the uncertainties in the design variables, also referred to as controllable variables, uncertainties in uncontrollable parameters are of special interest in systems design, [6]. Both serve as an input of a system model that is assumed to be certain in this work. In comparison to uncertainties concerning uncontrollable parameters, uncertainties in design variables often cannot be quantified mathematically. This is due to imprecise or incomplete knowledge which especially occurs during early phase of development. In order to tolerate maximal deviations in design variables, a robust design method was proposed in [4], and extended for example by [3] and [7]. Here, a design is sought that is the center of a maximal permissible neighborhood, which can be box-shaped, similar to the approach using solution spaces.
In this paper, worst-case solution spaces are introduced that also incorporate interval uncertainties in uncontrollable parameters. Moreover, the general idea of how solution spaces help to handle uncertainties in design variables is motivated and the connection to the robust design method mentioned above is illustrated. In chapter 2, basic definitions of systems design are given and occurring uncertainties are investigated. Chapter 3 discusses methods for systems design concerning uncertainties in design variables and leads to worst-case solution spaces. In chapter 4, standard and worst-case solution spaces are compared at a crash design problem.

2 Basics

2.1 Systems Design

A goal in complex systems or systems design is to find a design for which the system performs optimally. Here, a design can be expressed as a d-dimensional vector $\mathbf{x}$. The entries of $\mathbf{x}$ are the design variables x_i, $i = 1, \ldots, d$. Each x_i is technically lower bounded by $x^{\mathrm{l}}_{\mathrm{ds},i}$ and upper bounded by $x^{\mathrm{u}}_{\mathrm{ds},i}$. Thus, it holds componentwise $\mathbf{x}^{\mathrm{l}}_{\mathrm{ds}} \leq \mathbf{x} \leq \mathbf{x}^{\mathrm{u}}_{\mathrm{ds}}$ and $\Omega_{\mathrm{ds}} = [\mathbf{x}^{\mathrm{l}}_{\mathrm{ds}}, \mathbf{x}^{\mathrm{u}}_{\mathrm{ds}}]$, a multi-dimensional interval, can be defined as the design space. It is remarked that the same notations are used for all vector inequalities and multi-dimensional intervals in the following. Because the design can be chosen within Ω_{ds} by a decision maker,

the design variables are also called controllable variables.
Besides controllable variables, there are uncontrollable parameters, see [6]. Those are collected in a q-dimensional vector $\mathbf{p}$ and their values can neither be influenced by the decision maker anymore nor be influenced in general. Typical uncontrollable parameters include a priori chosen design variables, and system-specific or environmental parameters. Together, the design variables and the uncontrollable parameters form the inputs of the system. The outputs of the system are the responses which depend on the inputs and can be collected in a m-dimensional vector $\mathbf{z}$. A diagram for a system like this is visualized in Figure 1.

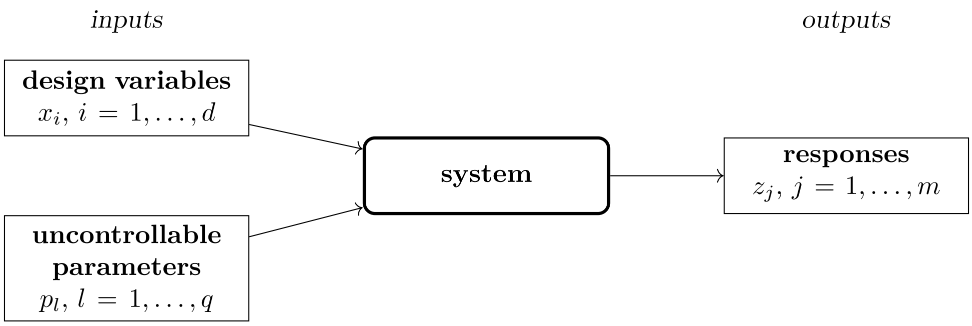

Figure 1: Diagram of a system with inputs and outputs

The mapping of the inputs to the outputs can be described by the system performance functions $\mathbf{f}$, where

$$\mathbf{z} = \mathbf{f}(\mathbf{x}, \mathbf{p}) \tag{1}$$

holds. In order to reach an optimal performance, some of the system responses must be minimized or maximized. Given these goals, an optimal design can be sought by using standard optimization methods which can also be multi-objective. However, as uncertainties can occur during systems development, it is possible that the resulting design causes a bad system performance. Therefore, uncertainty consideration has to be incorporated in the decision process for selecting a design.

2.2 Uncertainties

There are various ways to classify uncertainties, [1]. One is to differentiate from a system point of view. Uncertainties can be found in design variables, in uncontrollable parameters, and in the system model including its responses. Uncertainties in the design variables mainly occur because they cannot be realized accurately. This is for example due to manufacturing tolerances, or design variables are outputs of lower level design variables indeed. The same holds for uncontrollable parameters if they are a priori chosen design variables. Furthermore, there are also variations which follow for example from changing environmental or operating conditions. Uncertainties in the system model result from the approximation of real physical objects in order to use models. This includes also measuring errors when building the model. Other classifications of uncertainties can be done by means of their mathematical quantification or by distinguishing between aleatoric or epistemic type.
In early phase of system design, there is often not much information available on the uncertainties. This is mainly due to imprecise or incomplete knowledge which corresponds to the epistemic type. Therefore, a proper mathematical quantification is not possible in most cases. However, if it can be ensured that the realized values are bounded, the uncertainties can be at least modeled as interval uncertainties. In the following, the focus is put on uncertainties in design variables and in uncontrollable parameters. Similar to [6], it is assumed that the bounds of the uncontrollable parameters are known, i.e.

$$p_l \in [p_l^{\mathrm{l}}, p_l^{\mathrm{u}}] \tag{2}$$

with fixed p_l^{l} and p_l^{u} for $l = 1, \ldots, q$. The bounds of the design variables are unknown. Though, for given nominal values $\check{x}_i$, $i = 1, \ldots, d$ which correspond to the design values chosen by the designer, the real values are assumed to be symmetrically bounded around these nominal values, i.e.

$$x_i \in [\check{x}_i - \delta_i, \check{x}_i + \delta_i] \tag{3}$$

with $\delta_i \geq 0$, $\check{x}_i - \delta_i \geq x^{\mathrm{l}}_{\mathrm{ds},i}$, and $\check{x}_i + \delta_i \leq x^{\mathrm{u}}_{\mathrm{ds},i}$ for $i = 1, \dots, d$. As stated above, the values of δ_i, $i = 1, \dots, d$ are typically unknown. However, they can be also specified as maximal tolerances for the nominal values.

3 Solution Spaces

3.1 Motivation

In this section, the use of solution spaces for systems design and for handling the prescribed uncertain situation is motivated. Therefore, uncertainties in $\mathbf{p}$ are neglected at first. Furthermore, $\delta_i = \bar{\delta}$ for $i = 1, \dots, d$ is assumed and written as $\bar{\boldsymbol{\delta}} = (\bar{\delta}, \dots, \bar{\delta})$.
If only one performance function is given that shall be minimized, i.e. $m = 1$, the robust regularization $f_{\bar{\delta}}$ can be defined according to [5]. For given $\bar{\delta}$, it is

$$f_{\bar{\delta}}(\check{\mathbf{x}}, \mathbf{p}) = \sup\{f(\mathbf{x}, \mathbf{p}) : \ \mathbf{x} \in [\check{\mathbf{x}} - \bar{\boldsymbol{\delta}}, \check{\mathbf{x}} + \bar{\boldsymbol{\delta}}]\}, \tag{4}$$

which is a function of $\check{\mathbf{x}}$. Thus, by incorporating all possible realizations of $\mathbf{x}$, the worst case for the system performance is always regarded. Then, a minimization of the robust regularization yields a design $\check{\mathbf{x}}$ which can be considered as robust design. This is because the worst value of the system response for uncertain design variables is minimal.
In systems design it is not always necessary to minimize systems responses. Often, it is also sufficient if they do not exceed or fall below particular threshold values, [9]. If this is applicable, the problem can be tackled from the opposite side. As $\bar{\delta}$ is typically unknown, it can also be maximized under given constraints. In the following, multiple performance functions are regarded again, i.e. $m \geq 1$. Furthermore, thresholds $\mathbf{f}_{\mathrm{c}}(\mathbf{p})$ are given which upper bound the systems responses and depend on uncontrollable parameters. Hence, the set of all permissible designs, i.e. the designs within the design space which fulfill the constraints on their responses, are

$$\Omega_{\mathrm{c}} = \{\mathbf{x} \in \Omega_{\mathrm{ds}} : \ \mathbf{f}(\mathbf{x}, \mathbf{p}) \leq \mathbf{f}_{\mathrm{c}}(\mathbf{p})\}. \tag{5}$$

The set Ω_{c} is also called complete solution space. It shall be remarked that lower-bound thresholds can be considered in the same way. Therefore, they and their corresponding systems responses has to be multiplied by -1. In order to get nominal values for the design with maximal tolerances, the optimization problem reads

$$\begin{aligned} \underset{\check{\mathbf{x}}, \bar{\delta}}{\text{maximize}} \quad & \bar{\delta} \\ \text{subject to} \quad & \mathbf{x} \in \Omega_{\mathrm{ds}}, \ f(\mathbf{x}, \mathbf{p}) \leq f_c(\mathbf{p}) \quad \forall \mathbf{x} \in [\check{\mathbf{x}} - \bar{\boldsymbol{\delta}}, \check{\mathbf{x}} + \bar{\boldsymbol{\delta}}] \end{aligned} \tag{6}$$

with $\bar{\delta} \geq 0$ which was for example considered in [4]. Like before, its solution $(\check{\mathbf{x}}^*, \bar{\delta}^*)$ contains nominal values that can be regarded as robust design. Here, this design is forced into the center of Ω_{c}. Therefore, problem (6) is also called a design centering problem, [3].
As any $\mathbf{x} \in [\check{\mathbf{x}}^* - \bar{\boldsymbol{\delta}}^*, \check{\mathbf{x}}^* + \bar{\boldsymbol{\delta}}^*]$ with $\bar{\boldsymbol{\delta}}^* = (\bar{\delta}^*, \dots, \bar{\delta}^*)$ is a permissible design, the set

$$\Omega^* = [\check{\mathbf{x}}^* - \bar{\boldsymbol{\delta}}^*, \check{\mathbf{x}}^* + \bar{\boldsymbol{\delta}}^*] \tag{7}$$

is a box-shaped solution space within the complete solution space. Further, the solution space Ω^* can be used for systems design itself while offering some advantages which are discussed in the following. The method was first introduced by [9].

3.2 The standard and the worst case

In Figure 2, the solution space Ω^* is visualized.

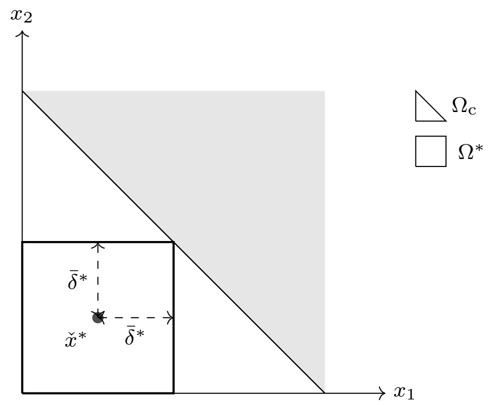

Figure 2: Solution space Ω^* defined by the robust design $\check{\mathbf{x}}^*$ and the maximal tolerance $\bar{\delta}^*$

The solution space Ω^*, can be used for systems design without having to regard the design space or system responses separately. Given Ω^*, the decision maker can choose a design in form of nominal values, i.e. $\check{\mathbf{x}}$ within Ω^*, by regarding the following situations: If the real values of $\bar{\delta}$ are unknown, the decision maker should always select $\check{\mathbf{x}}$, which is the center of the box-shaped solution space, in order to tolerate maximal variations of that design. If the real values of $\bar{\delta}$ are known however and $\bar{\delta} \leq \bar{\delta}^*$ holds, he can select $\check{\mathbf{x}}$ from $[\check{\mathbf{x}}^* - (\bar{\boldsymbol{\delta}}^* - \bar{\boldsymbol{\delta}}), \check{\mathbf{x}}^* + (\bar{\boldsymbol{\delta}}^* - \bar{\boldsymbol{\delta}})]$ and all variations from that design are guaranteed to be permissible. This can be done for example by minimizing the robust regularization or by solving another optimization problem. Otherwise, there is no design where all variations from that design can be guaranteed to be permissible.

Nevertheless, there are no benefits from choosing a design from the box-shaped solution space compared to the complete solution space so far. Instead, some permissible design possibilities from the complete solution space are lost. Though, the major advantages of using Ω^* is that decoupled decisions for choosing design variables are enabled. Each $\check{x}_i$ can be chosen within $[\check{x}_i^* - (\bar{\delta}^* - \bar{\delta}_i), \check{x}_i^* + (\bar{\delta}^* - \bar{\delta}_i)]$ independently without having to incorporate the other decisions regarding the design variables for $i = 1, \ldots, d$. Thus, the decisions also do not have to be met simultaneously and single decisions can be easily adapted at a later stage. Together with the property that the design which tolerates the largest variations of all designs in Ω_c is the center of Ω^*, providing box-shaped solution spaces can be a useful tool for systems design.

However, from the perspective of uncertainties, solution spaces only account for uncertainties in design variables. In this paper, uncertainties in the uncontrollable parameters, like described in (2), are also incorporated. In doing so, the optimization problem (6) becomes

$$\begin{aligned} &\underset{\check{\mathbf{x}},\bar{\delta}}{\text{maximize}} && \bar{\delta} \\ &\text{subject to} && \mathbf{x} \in \Omega_\mathrm{ds},\ f(\mathbf{x},\mathbf{p}) \leq f_c(\mathbf{p}) \quad \forall \mathbf{x} \in [\check{\mathbf{x}} - \bar{\boldsymbol{\delta}}, \check{\mathbf{x}} + \bar{\boldsymbol{\delta}}],\ \forall \mathbf{p} \in [\mathbf{p}^\mathrm{l}, \mathbf{p}^\mathrm{u}] \end{aligned} \tag{8}$$

where $\bar{\delta} \geq 0$ must be guaranteed. The solution of this problem $(\check{\mathbf{x}}^\dagger, \bar{\delta}^\dagger)$ builds up the worst case box-shaped solution space $\Omega^\dagger$. Here, the constraints $f(\mathbf{x},\mathbf{p}) \leq f_c(\mathbf{p})$ can be considered as hard constraints because the designs in $\Omega^\dagger$ must fulfill them for every realization of $\mathbf{p} \in [\mathbf{p}^\mathrm{l}, \mathbf{p}^\mathrm{u}]$. Selecting a design $\check{\mathbf{x}}$ within $\Omega^\dagger$ can be done similarly to the case which does not consider uncertainties in $\mathbf{p}$. As this procedure is further restricted due to uncertainties in the design variables, the overall design process becomes very conservative.

3.3 Reformulation of the problem

The optimization problem (8) is not easy to solve for arbitrary system performance functions $\mathbf{f}$. In the following, the case where f_j is a linear or rather affine function of $\mathbf{x}$ with

$$f_j(\mathbf{x},\mathbf{p}) - f_{\mathrm{c},j}(\mathbf{p}) = \mathbf{a}_j^\mathrm{T}(\mathbf{p})\mathbf{x} - b_j(\mathbf{p}) \tag{9}$$

for $j = 1, \ldots, m$ is considered. Here, $\mathbf{a}_j(\mathbf{p})$ is a d-dimensional vector with $\mathbf{a}_j(\mathbf{p}) = (a_{j,1}(\mathbf{p}), \ldots, a_{j,d}(\mathbf{p}))$, and $b_i(\mathbf{p})$ is a scalar. In total, the inequalities $f_j(\mathbf{x},\mathbf{p}) \leq f_{\mathrm{c},j}(\mathbf{p})$ are a system of linear inequalities in the form $\mathbf{A}(\mathbf{p})\mathbf{x} \leq \mathbf{b}(\mathbf{p})$ for fixed $\mathbf{p}$, where the rows of $\mathbf{A}(\mathbf{p})$ and $\mathbf{b}(\mathbf{p})$ are formed by $\mathbf{a}_j(\mathbf{p})$ and $b_j(\mathbf{p})$

respectively. With equation (9), problem (8) becomes

$$\begin{array}{ll} \underset{\check{\mathbf{x}},\bar{\delta}}{\text{maximize}} & \bar{\delta} \\ \text{subject to} & \mathbf{x} \in \Omega_{\text{ds}},\ \mathbf{a}_j^{\text{T}}(\mathbf{p})\mathbf{x} \leq b_j(\mathbf{p}) \quad \forall \mathbf{x} \in [\check{\mathbf{x}} - \bar{\boldsymbol{\delta}}, \check{\mathbf{x}} + \bar{\boldsymbol{\delta}}],\ \forall \mathbf{p} \in [\mathbf{p}^{\text{l}}, \mathbf{p}^{\text{u}}],\ j = 1, \ldots, m, \end{array} \tag{10}$$

where $\bar{\delta} \geq 0$ has to be ensured. Here, the inequality $\mathbf{a}_j^{\text{T}}(\mathbf{p})\mathbf{x} \leq b_j(\mathbf{p})$ is fulfilled for all $\mathbf{x} \in [\check{\mathbf{x}} - \bar{\boldsymbol{\delta}}, \check{\mathbf{x}} + \bar{\boldsymbol{\delta}}]$ and all $\mathbf{p} \in [\mathbf{p}^{\text{l}}, \mathbf{p}^{\text{u}}]$ if and only if it is $\sup\{\mathbf{a}_j^{\text{T}}(\mathbf{p})\mathbf{x} :\ \mathbf{x} \in [\check{\mathbf{x}} - \bar{\boldsymbol{\delta}}, \check{\mathbf{x}} + \bar{\boldsymbol{\delta}}],\ \mathbf{p} \in [\mathbf{p}^{\text{l}}, \mathbf{p}^{\text{u}}]\} \leq \inf\{b_j(\mathbf{p}) :\ \mathbf{p} \in [\mathbf{p}^{\text{l}}, \mathbf{p}^{\text{u}}]\}$. Given that $x^{\text{l}}_{\text{ds},i} \geq 0$, $i = 1, \ldots, d$, holds, then it is

$$\sup\{a_{j,i}(\mathbf{p})x_i :\ \mathbf{p} \in [\mathbf{p}^{\text{l}}, \mathbf{p}^{\text{u}}],\ x_i \in [\check{x}_i - \bar{\delta}, \check{x}_i + \bar{\delta}]\} = \sup\{a_{j,i}(\mathbf{p}) : \mathbf{p} \in [\mathbf{p}^{\text{l}}, \mathbf{p}^{\text{u}}]\}(\check{x}_i + w_{j,i}\bar{\delta}) \tag{11}$$

with

$$w_{j,i} = \begin{cases} -1 & \text{for } \sup\{a_{j,i}(\mathbf{p}) : \mathbf{p} \in [\mathbf{p}^{\text{l}}, \mathbf{p}^{\text{u}}]\} \leq 0, \\ \ \ 1 & \text{for } \sup\{a_{j,i}(\mathbf{p}) : \mathbf{p} \in [\mathbf{p}^{\text{l}}, \mathbf{p}^{\text{u}}]\} > 0 \end{cases} \tag{12}$$

for $i = 1, \ldots, d$, $j = 1, \ldots, m$. Moreover, if there is a $\mathbf{p}^{*,j} \in [\mathbf{p}^{\text{l}}, \mathbf{p}^{\text{u}}]$ for each j that maximizes $a_{j,i}(\mathbf{p})$ for $i = 1, \ldots, d$ and minimizes $b_j(\mathbf{p})$ simultaneously, then problem (10) can be reformulated to

$$\begin{array}{ll} \underset{\check{\mathbf{x}},\bar{\delta}}{\text{maximize}} & \bar{\delta} \\ \text{subject to} & -\bar{\delta} \leq 0,\ -\check{\mathbf{x}} + \bar{\boldsymbol{\delta}} \leq -\mathbf{x}^{l}_{\text{ds}},\ \check{\mathbf{x}} + \bar{\boldsymbol{\delta}} \leq \mathbf{x}^{u}_{\text{ds}}, \\ & \mathbf{a}_j^{\text{T}}(\mathbf{p}^{*,j})\check{\mathbf{x}} + \mathbf{a}_j^{\text{T}}(\mathbf{p}^{*,j})\mathbf{W}_j\bar{\boldsymbol{\delta}} \leq b_j(\mathbf{p}^{*,j}),\ j = 1, \ldots, m, \end{array} \tag{13}$$

where $\mathbf{W}_j$ is a diagonal $d \times d$-matrix for that the i^{th} entry on the diagonal is given by $w_{j,i}$ from equation (12). A proof for this can be done similar to [3]. Problem (13) is a linear optimization problem and can be solved numerically by standard linear optimization techniques.

4 Application to Crash Design

4.1 Systems Specifications

A vehicle that has to perform in a frontal crash can be considered as a systems design problem. Here, significant performance measures which are responses of the system can be the maximal acceleration, the energy absorption, and the order of deformation of the components. For a given vehicle structure and a deformation in one direction, all of them can be calculated from the corresponding force-deformation characteristics of its components. Here, it is assumed that the force-deformation characteristics can be designed directly where their degrees of freedom are used as design variables. In general however, they are only responses of lower level design variables, like material properties. Therefore, an appropriate calculation of the lower level design variables must follow in a second step which is not considered in this paper.

In the following, the focus is put on a scenario where the vehicle is driven against a rigid wall at full overlap with an initial velocity v_0, like done in crash tests. Instead of optimizing the system responses, requirements in the form of threshold values are set. These requirements contain the bounding of the maximal acceleration by a critical a_{c}, the complete absorption of the impact energy, and that the deformation of the components starts at the front of the vehicle.

As an example for a crash design problem, a front structure of a vehicle, taken from [2], is shown in Figure 3 and considered. This structure was obtained from the geometry of a system by assuming a discrete vehicle mass distribution, given by m_1, m_2, and m_3, and by mapping the coordinates of the geometry to coordinates of simultaneous deformation. In addition, it is assumed here that every component will deform only partially, given as a combined sum of s_1, s_2, and s_3, before it behaves rigid. These rigid parts were also removed from the geometry in Figure 3.

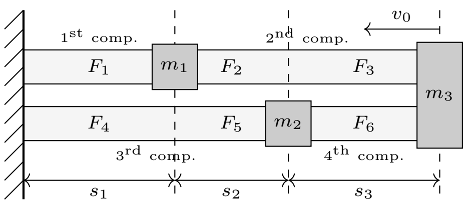

Figure 3: Front structure of a vehicle with four components, three masses, two load paths, and three sections, compare [2]

There are four components and therefore four force-deformation characteristics. In every section, they are modeled as piece-wise constant. Thus, there are six degrees of freedom in total, meaning six design variables. One design variable belongs to each the first (F_1) and the forth (F_6) and two to each the second (F_2 and F_3) and the third (F_4 and F_5) component. Every F_i is lower bounded by $F^{\mathrm{l}}_{\mathrm{ds},i} = 0\,\mathrm{kN}$ and upper bounded by $F^{\mathrm{u}}_{\mathrm{ds},i} = 300\,\mathrm{kN}$ for $i = 1, \dots, 6$ what forms the design space Ω_{ds}.
In order to keep the maximal acceleration below the critical a_{c}, the sum of deformation forces of simultaneously deforming parts must be bounded, which is stated by three linear inequalities for the given structure. These inequalities are composed by the columns 1-3 of $\mathbf{A}(\mathbf{p})$ in equation (15) and of $\mathbf{b}(\mathbf{p})$ in equation (16). To absorb the complete impact energy, the sum of integrals of the component's deformation force over their deformation length must be greater than the impact energy itself. As the deformation forces are constant in every section this constraint is also a linear inequality which is composed by the columns 4 of $\mathbf{A}(\mathbf{p})$ in equation (15) and of $\mathbf{b}(\mathbf{p})$ in equation (16). To ensure that the deformation of the components starts at the front of the vehicle, the deformation force in every section of a previous component must be smaller than the force which is necessary to start the deformation of a subsequent component. Again, these requirements can be expressed as linear inequalities which are composed by the columns 5-7 of $\mathbf{A}(\mathbf{p})$ in equation (15) and of $\mathbf{b}(\mathbf{p})$ in equation (16).
In total, constraints on the system responses are given by

$$\mathbf{A}(\mathbf{p})(F_1, \dots, F_6)^{\mathrm{T}} \leq \mathbf{b}(\mathbf{p}) \tag{14}$$

with

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -\frac{s_1}{m_1+m_2+m_3} & -\frac{s_2}{m_2+m_3} & -\frac{s_3}{m_3} & -\frac{s_1}{m_1+m_2+m_3} & -\frac{s_2}{m_2+m_3} & -\frac{s_3}{m_3} \\ 1-\frac{m_1}{m_1+m_2+m_3} & -1 & 0 & -\frac{m_1}{m_1+m_2+m_3} & 0 & 0 \\ -\frac{m_2}{m_1+m_2+m_3} & 0 & 0 & 1-\frac{m_2}{m_1+m_2+m_3} & 0 & -1 \\ 0 & -\frac{m_2}{m_2+m_3} & 0 & 0 & 1-\frac{m_2}{m_2+m_3} & -1 \end{pmatrix} \tag{15}$$

and

$$\mathbf{b}(\mathbf{p}) = \begin{pmatrix} (m_1+m_2+m_3)a_{\mathrm{c}} & (m_2+m_3)a_{\mathrm{c}} & m_3 a_{\mathrm{c}} & -\frac{1}{2}(v_0)^2 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \tag{16}$$

see [2]. Therefore, the system performance functions together with their threshold values can be also written in the form of equation (9). The uncontrollable parameters are collected in the vector $\mathbf{p}$ with $\mathbf{p} = (s_1, s_2, s_3, m_1, m_2, m_3, v_0, a_{\mathrm{c}})$. These are the section lengths s_1, s_2, s_3, the masses m_1, m_2, m_3, the initial velocity of the vehicle v_0, and the critical acceleration a_{c}. Here, the intervals where their real values can be found are given in table 1.

Uncontrollable parameter	Interval of the real values
s_1	$[0.19\,\mathrm{m}, 0.21\,\mathrm{m}]$
s_2	$[0.14\,\mathrm{m}, 0.16\,\mathrm{m}]$
s_3	$[0.19\,\mathrm{m}, 0.21\,\mathrm{m}]$
m_1	$[95\,\mathrm{kg}, 105\,\mathrm{kg}]$
m_2	$[145\,\mathrm{kg}, 155\,\mathrm{kg}]$
m_3	$[1150\,\mathrm{kg}, 1250\,\mathrm{kg}]$
v_0	$[15.5\,\frac{\mathrm{m}}{\mathrm{s}}, 15.6\,\frac{\mathrm{m}}{\mathrm{s}}]$
a_{c}	$[290\,\frac{\mathrm{m}}{\mathrm{s}^2}, 310\,\frac{\mathrm{m}}{\mathrm{s}^2}]$

Table 1: Uncertainty modelling of the uncontrollable parameters

As there exists a $\mathbf{p}^{*,j} \in [\mathbf{p}^{\mathrm{l}}, \mathbf{p}^{\mathrm{u}}]$ that maximizes the entries of the j^{th} column of $\mathbf{A}(\mathbf{p})$ and minimizes the j^{th} entry of $\mathbf{b}(\mathbf{p})$ for $j = 1, \dots, m$, the crash design problem to find worst-case box-shaped solution space can be formulated in the form of problem (13) and be solved accordingly.

4.2 Solution Spaces for Crash Design

In Figure 4, the worst case box-shaped solution space $\Omega^{\dagger}$ for the crash design problem is visualized.

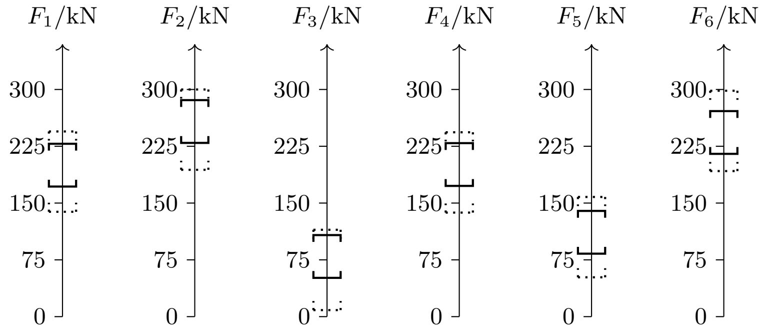

Figure 4: Intervals that form the worst case solution space $\Omega^\dagger$ (solid lines) with $\mathbf{p} \in [\mathbf{p}^\mathrm{l}, \mathbf{p}^\mathrm{u}]$ and the standard solution space Ω^* (dotted lines) with $\mathbf{p} = \frac{1}{2}(\mathbf{p}^\mathrm{l} + \mathbf{p}^\mathrm{u})$ for the crash design problem

In addition to the worst case solution space, also the standard case with $\mathbf{p} = \frac{1}{2}(\mathbf{p}^\mathrm{l} + \mathbf{p}^\mathrm{u})$ is shown. As $\frac{1}{2}(\mathbf{p}^\mathrm{l} + \mathbf{p}^\mathrm{u})$ is contained in $[\mathbf{p}^\mathrm{l}, \mathbf{p}^\mathrm{u}]$, the complete solution space that accounts for all $\mathbf{p} \in [\mathbf{p}^\mathrm{l}, \mathbf{p}^\mathrm{u}]$ is contained in the one that accounts only for $\mathbf{p} = \frac{1}{2}(\mathbf{p}^\mathrm{l} + \mathbf{p}^\mathrm{u})$. Therefore, the sizes of the intervals that form $\Omega^\dagger$ are smaller or equal to the ones that form Ω^*. For the crash design problem, these intervals are contained in each other where the interior intervals are oriented towards the upper bounds of the exterior intervals, see Figure 4. This can be explained by the influence of the uncertainties in uncontrollable parameters on the constraints that build the solution space. Here, the influence on the energy absorption which is related to small design variable values is greater than the influence on the maximal acceleration which concerns large design variable values.

5 Conclusions

In this paper, worst-case box-shaped solution spaces were introduced as a method for systems design. These solution spaces consist of maximal intervals for the design variables within these, all requirements on the system responses are satisfied. Furthermore, a decoupled selection of the design variables is enabled by using solution spaces. Compared to standard solution spaces not only uncertainties in controllable design variable are considered in this method, but also uncertainties in uncontrollable parameters which were given as intervals here.

Additionally, it was shown how worst-case solution spaces can be calculated for affine performance functions. Applied to a crash design problem, the influence of uncertainties in uncontrollable parameters on the solution spaces could be illustrated. For further research, the focus should not be only limited to affine performance functions and interval uncertainties for uncontrollable parameters in order to extend the methods applicability to systems design.

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