P-solutions of linear and nonlinear interval parametric systems and applications

L. Kolev

Theoretical Electrical Eng., TU-Sofia, 1000 Sofia, Bulgaria,

E-mail: lkolev@tu-sofia.bg

Outline

- Introduction
- Properties of P-solutions
- Methods for determining P-solutions
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1. Introduction

Consider the following parametric systems of decreasing generality and complexity:

- nonlinear interval parametric (NLIP) system

$$f(x, p) = 0$$

$$f_i(x_1,..x_n; p_1,..p_m) = 0, \quad p_j \in \boldsymbol{p}_j = [-1, 1], \quad i = 1,..,n \qquad (1.10)$$

x_j are the variables and p_j are the parameters

- linear interval parametric (LIP) system

$$f(x, p) = A(p)x - b(p) = 0 \quad , \quad p \in \boldsymbol{p} \qquad (1.11)$$

- nonlinear parametric dependences (NLPD) systems

- linear parametric dependence (LPD) systems, where

$$a_{ij}(p) = \alpha_{ij} + \sum_{\mu=1}^{m} a_{ij\mu} p_{\mu} \,, \qquad b_i(p) = \beta_i + \sum_{\mu=1}^{m} \beta_{i\mu} p_{\mu} \qquad (1.11b)$$

We need the concept of the solution set Σ of (1.10) or (1.11) defined as follows

$$\Sigma = \{x : f(x, p) = 0,\ p \in \boldsymbol{p}\} \qquad (1.12)$$

Known "interval solutions" to (1.10) or (1.11):(i) interval hull (IH) solution $\boldsymbol{x}^*$: the smallest interval vector containing Σ; (ii) outer interval (OI) solution $\boldsymbol{x}$: any interval vector enclosing $\boldsymbol{x}^*$, i.e. $\boldsymbol{x}^* \subseteq \boldsymbol{x}$ and (iii) inner estimation of the hull (IEH) solution ξ : an interval vector such that $\xi \subseteq \boldsymbol{x}^*$.

Recently, a new type of solution to LPD systems (1.11), (1.11b) has been introduced [Kolev 2014] which is of the following parametric form

$$\boldsymbol{x}(p) = Lp + \boldsymbol{a}\,, \quad p \in \boldsymbol{p} \tag{1.13}$$

where L is a real $n \times m$ matrix whereas $\boldsymbol{a}$ is an interval vector. It is called a parameterized solution (p-solution) of (1.11). The p-solution is an outward linear approximation of Σ.

Iterative methods for determining $\boldsymbol{x}(p)$ were suggested for the LPD case in [Kolev 2014,2016 Rel. Comp.]; a simple direct method was proposed in [Kolev 2016 JACM]. Also the NLPD case was considered in [Skalna and Hladik 2017].

2. Properties of P-solutions

Let $x(p)$ be a p-solution of system (1.10). Interval hull $\boldsymbol{x}(\boldsymbol{p})$ of $x(p)$ - the smallest interval containing $x(p)$.

- **Property P1**. The interval hull $\boldsymbol{x}(\boldsymbol{p})$ of $x(p)$ is an outer interval solution $\boldsymbol{x}$ of (1.10), i.e. $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p})$.

Consider the ith component x_i^* of the IH solution x^* to (1.10) and the ith component $x_i(p)$ of $x(p)$

$$x_i(p) = \breve{x}_i + L_i p + \hat{x}_i[-1,1], \quad p \in \boldsymbol{p} \qquad (3.1)$$

($\breve{x}_i$, $\hat{x}_i$ - centre and radius, L_i - ith row of L) . Compute the ends

$$\underline{x}_i = \breve{x}_i - \sum_j |L_j| - \hat{x}_i \,, \quad \overline{x}_i = \breve{x}_i + \sum_j |L_j| + \hat{x}_i \qquad (3.2)$$

Introduce the intervals:

$$\boldsymbol{e}_i^{(\mathrm{l})} = [\underline{x}_i\,,\, \underline{x}_i + 2\,\hat{x}_i\,],\quad \boldsymbol{e}_i^{(\mathrm{u})} = [\overline{x}_i\,,\, \overline{x} - 2\,\hat{x}_i\,] \tag{3.4}$$

- **Property P2**. The intervals (3.4) contain the endpoints of $\boldsymbol{x}_i^*$, i.e.

$$\underline{x}_i^* \in \boldsymbol{e}_i^{(\mathrm{l})},\quad \overline{x}_i^* \in \boldsymbol{e}_i^{(\mathrm{u})} \tag{3.5}$$

Thus, the intervals (3.4) provide two-sided bounds on the ends of $\boldsymbol{x}_i^*$

- **Property P3**. Introduce the intervals

$$\xi_i = \begin{cases} [\overline{e}_i^{(l)},\ \underline{e}_i^{(u)}], & \mathit{if}\ \overline{e}_i^{(l)} < \underline{e}_i^{(u)} \\ \text{empty interval}, & \mathit{otherwise} \end{cases} \tag{3.6}$$

Then ξ_i determines the ith component of the IEH solution of (1.10).

3. Methods for determining a *p*-solution

All known methods refer to LIP systems and have polynomial time complexsity. Here we only mention the

3.1. *Direct method of [Kolev 2016] (Method K)*

which requires roughly $N_k = n^4 m$ arithmetic operations

Comparison with method of [Skalna 2006] (Method S_k)

Methods K and S_k have the same complexity. However, unlike method K, method S_k has only Property P1.

3.2. *A new method*

It is applicable to the general type of NLIP systems

$$f(x, p) = 0, \quad y \in \mathbf{p} \tag{3.32}$$

Computational scheme.

1). An outer interval solution $\boldsymbol{x}^0$ is computed (using some standard method).

2). The variables $\boldsymbol{x}^0$ and $\boldsymbol{y}$ in (3.32) are then treated as independent, so (3.32) is replaced by

$$f(x,p)=0\,,\quad x\in\boldsymbol{x}^0\,,\quad p\in\boldsymbol{p} \tag{3.34}$$

3). The variables $\boldsymbol{x}_i$ are written in normalized form

$$x_i=\check{x}_i+\hat{x}_i q_i\,,\quad q_i\in\boldsymbol{q}_i=[-1,1] \tag{3.35a}$$

Using (3.35a), (3.34) is modified to

$$f(q,p)=0 \tag{3.37a}$$

4). Using affine arithmetic (AA) the nonlinear system (3.37a) is enclosed by the LIP system.

$$\boldsymbol{l}(q,p)=\breve{l}+A^{q}q+A^{p}p+\hat{l}[-1,1]=0,\quad q\in\boldsymbol{q},\quad p\in\boldsymbol{p} \tag{3.37b}$$

where A^q and A^p are matrices of size $n\times n$ and $n\times m$

5). Solve for

$$q=\breve{q}+Lp+\hat{q}[-1,1],\quad p\in\boldsymbol{p} \tag{3.38}$$

6). Using (3.35a) and q we compute

$$\boldsymbol{x}(p)=\breve{x}+Lp+\hat{x}[-1,1],\quad p\in\boldsymbol{p} \tag{3.40}$$

7). Let $\boldsymbol{x}$ be the interval hull of (3.40)

If

$$\boldsymbol{x}\subseteq\boldsymbol{x}^{0} \tag{3.40a}$$

then $\boldsymbol{x}(p)$ obtained by (3.35a) to (3.40) is a parameterized solution of (3.32)

If $\boldsymbol{x}$ is narrower than $\boldsymbol{x}^0$ and the reduction is larger than a threshold ε_r, then $\boldsymbol{x}$ is renamed $\boldsymbol{x}^0$ and a new iteration can be resumed from (3.35) until some stopping criterion is met.

Numerical example with $n = 2$ and $m = 2$

The method was programmed in MATLAB environment using the toolbox IntLab.8 [Rump 1999]. The AA arithmetic was implemented by the *affari* toolbox [Rump and Kashiwagi 2015]

The system (3.34) is

$$f_1(x, y) = -y_1 x_1 + x_2^2 = 0, \qquad (3.46a)$$

$$f_2(x, y) = y_2 f_{21}(x_1) + x_2^2 = 0, \quad f_{21}(x_1) = x_1(0.5x_1 - 1) - 1, \qquad (3.46b)$$

$$y \in \boldsymbol{y}, \quad x \in \boldsymbol{x}^0 \qquad (3.46c)$$

$$y_1 \in \boldsymbol{y}_1 = [15.8, 16.2],\ y_2 \in \boldsymbol{y}_2 = [18, 19.6]$$

$$\boldsymbol{x}_1^0 = [1.2, 1.7], \quad \boldsymbol{x}_2^0 = [-5.8, -4.7]$$

Determining a p-solution

The related system (3.37b) is

$$\breve{l} = \begin{pmatrix} -4.5137 \\ -1.7110 \end{pmatrix}, \quad A^q = \begin{pmatrix} -4.000 & -5.775 \\ 2.115 & -5.775 \end{pmatrix}, \quad A^p = \begin{pmatrix} 0.2900 & 0 \\ 0 & 1.1065 \end{pmatrix}, \quad \hat{l} = \begin{pmatrix} 0.20125 \\ 0.54750 \end{pmatrix}$$

Hence

$$\boldsymbol{x}(p) = \breve{x} + Lp + \hat{x}[-1, 1], \quad p \in \boldsymbol{p} \tag{3.47}$$

$$\check{x} = \begin{pmatrix} 1.5645850 \\ -4.9947248 \end{pmatrix}, \quad L = \begin{pmatrix} -0.0118561 & 0.0523712 \\ -0.0095526 & -0.0689328 \end{pmatrix}, \quad \hat{x} = \begin{pmatrix} 0.0877044 \\ 0.1192227 \end{pmatrix} \tag{3.47a}$$

Finally

$$\boldsymbol{x} = \begin{pmatrix} [1.4768806, \ \ 1.6243233] \\ [-5.1139475, -4.8755021] \end{pmatrix} \tag{3.48}$$

and it is seen that $\boldsymbol{x} \subset \boldsymbol{x}^0$

Thus, the LIP form (3.47) determines the *p*-solution sought.

The subsequent iterations of method M1 yield better enclosures.

Convergence is reached in $k = 4$ iterations (relative reduction of $\hat{x}_i^0$ to $\hat{x}_i$ is less 1%)

Property 2. Determining the two-sided bound

We illustrate with $\boldsymbol{e}_2^l = [\underline{e}_2^l, \overline{e}_2^l]$. For k=4

$$\boldsymbol{e}_2^l = [-5.1139, -4.8755]$$

The lower end $\underline{x}_2^*$ of the IH solution $\boldsymbol{x}_2^*$ is approximately

$$\underline{x}_2^* = -5.0884$$

and it is seen that indeed

$$\underline{x}_2^* \in \boldsymbol{e}_2^1$$

4. A class of global optimization problems

Find the global minimum

$$f_0^* = \min f_0(x, p) \qquad (4.1a)$$

$$f_i(x, p) = 0, \quad p \in \boldsymbol{p}, \quad i = 1, 2, \ldots, n \qquad (4.1b)$$

The p-solution of (4.1b) – effective to solve (4.1).

Each individual problem is set up by specifying the functions $f_0(x, p)$ and $f(x, p)$. For instance:

Linear programing (LP) problem

$$f_0(p) = c^T(p)x(p)$$

where the constraint is the LIP system $A(p)x = b(p)$ [Kolev 2016]

Numerical example

$$c^T = (1, 1, 1)$$

If $c = \pm e_k$ (e_k is the k-th column of the identity matrix) the LP problem determines the lower/upper end of $\boldsymbol{x}_k^*$

A method for solving such problems is available [Kolev and Skalna 2017] which is based on expressing

$$\boldsymbol{f}_0(p) = \sum_i \boldsymbol{x}_i(p), \quad p \in \boldsymbol{p}$$

Numerical evidence shows that it is superior to standard methods wrt:

- enclosure efficiency: tightness of the approximation of the solution
- computational cost;
- applicability radius [Kolev 2014, 2016]: largest radius of the box within which the respective method is still applicable

5. Applications

- *Worst-case tolerance analysis for electric circuits* [Kolev 2002]
  - *Direct current circuits*
  - *Alternating current circuits*

- *Power consumption analysis* [Kolev 2011, 2013]
- *Truss analysis* [Muhanna et. Al. 2004, 2005; Popova 2006]

Each of the above problems consists in determining or bounding a component x_k^* of the corresponding hull solution x^*. The latter problem can be solved as corresponding simple interval linear programming ($c = \pm e_k$) using the approach of Section 4.

- *Eigenvalue range determination*

The standard method [Kolev 2010] can be improved using corresponding *p*-solutions

6. Conclusions

- The new methods employing p-solutions are better than their standard counterparts.
- NLIP problems can now be addressed using the new method of Section 3.2.
- Further numerical evidence would be welcome.
- It is hoped that the present survey will help promote the use of the new *p*-solution approach for solving various engineering problems involving uncertainties and risks.

Acknowledgement. The author is grateful to Prof. Robert Mullen for agreeing to give this presentation.