Estimating rare-event probabilities without data

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Engineers cannot always get data

- New systems may have no performance history
  - spacecraft of new design or in a new environment
  - biological control strategies using novel genetic constructs that have never existed before
- Two ways to estimate probabilities without data
  - disaggregation into parts whose probabilities are easier to estimate (i.e., breaking it into subproblems)
  - expert elicitation (i.e., guessing)

Is there no uncertainty?

- What is the uncertainty in estimates like
  - "1 in 10^7",
  - "about 1 in 1000", or
  - "never been seen in over 100 years of observation"?
- How should this uncertainty be captured and projected in computations?

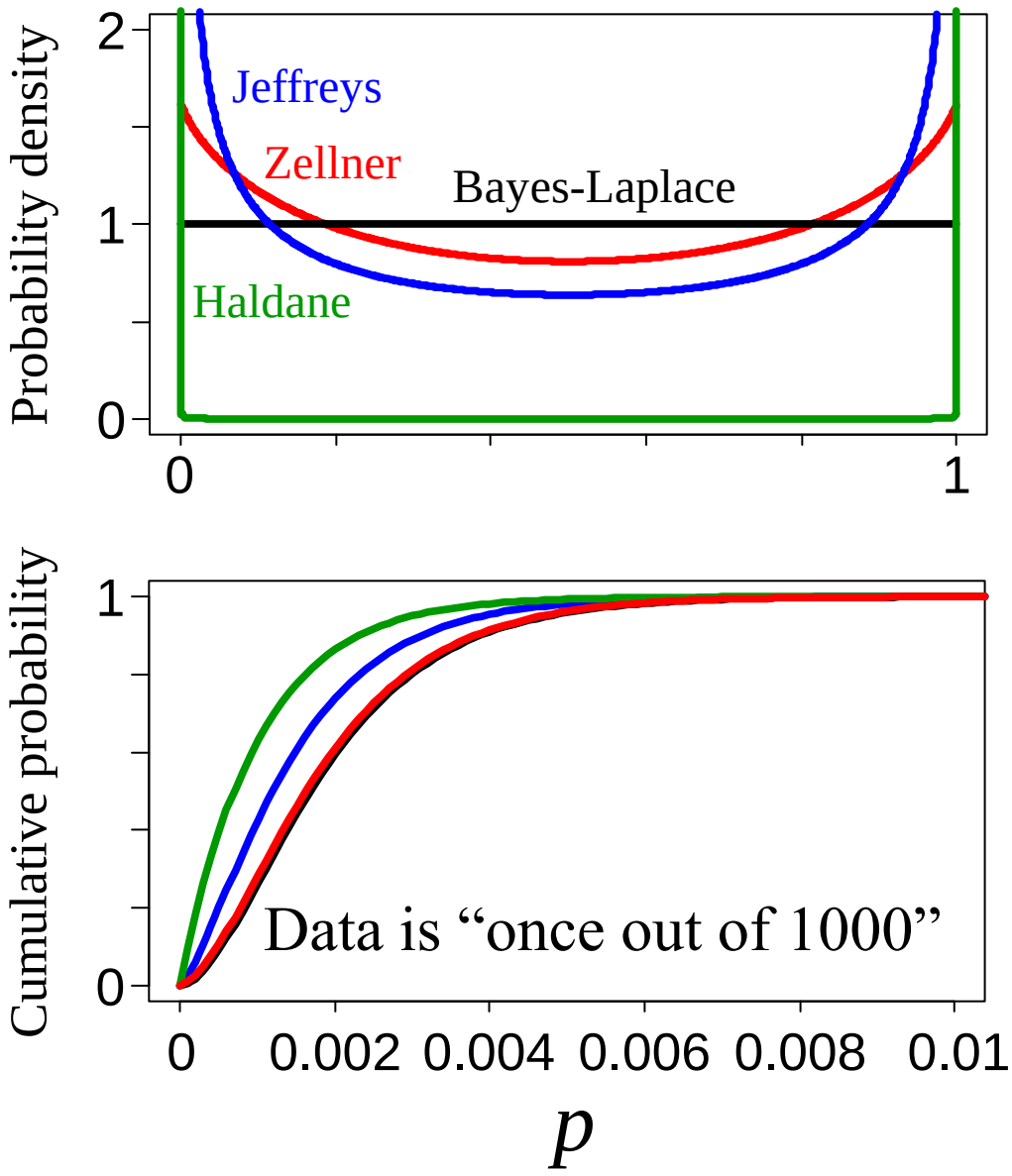
Rare events

- Often the driving concern in analyses
- Typically big consequences
- Hardly ever characterized by good data
- Perhaps never seen, or seen only once

Random sample data

- ML says *p* is zero for never-seen events
  - Nobody believes this is a reasonable estimate
- Bayesian estimator is more reasonable…

Just kidding...



Means

0.001	Haldane
0.0015	Jeffreys
0.001986	Zellner
0.001996	Bayes-Laplace

Random sample data

- ML says *p* is zero for never-seen events
  - Nobody believes this is a reasonable estimate
- The Bayesian estimator is many things
  - 'Reasonable' only in that *it's whatever you want*
- Modern estimators
  - Imprecise beta (Dirichlet) models
  - Confidence structures

Confidence structure (c-box)

- P-box-shaped estimator of a (fixed) parameter
- Gives confidence interval at *any* confidence level

Estimators

- Point estimates (e.g., sample mean)
- Interval estimates (e.g., confidence intervals)
- Distributional estimates (Bayesian posteriors)
- P-box-shaped estimates (e.g., c-boxes)

Probability of rare event

- Inference about probability from binary data, k successes out of n trials

$$p \sim [\text{beta}(k, n-k+1), \text{beta}(k+1, n-k)]$$

Notation extends the use of tilda

- Identical to Walley's Imprecise Beta Model with $s=1$, but needs no prior

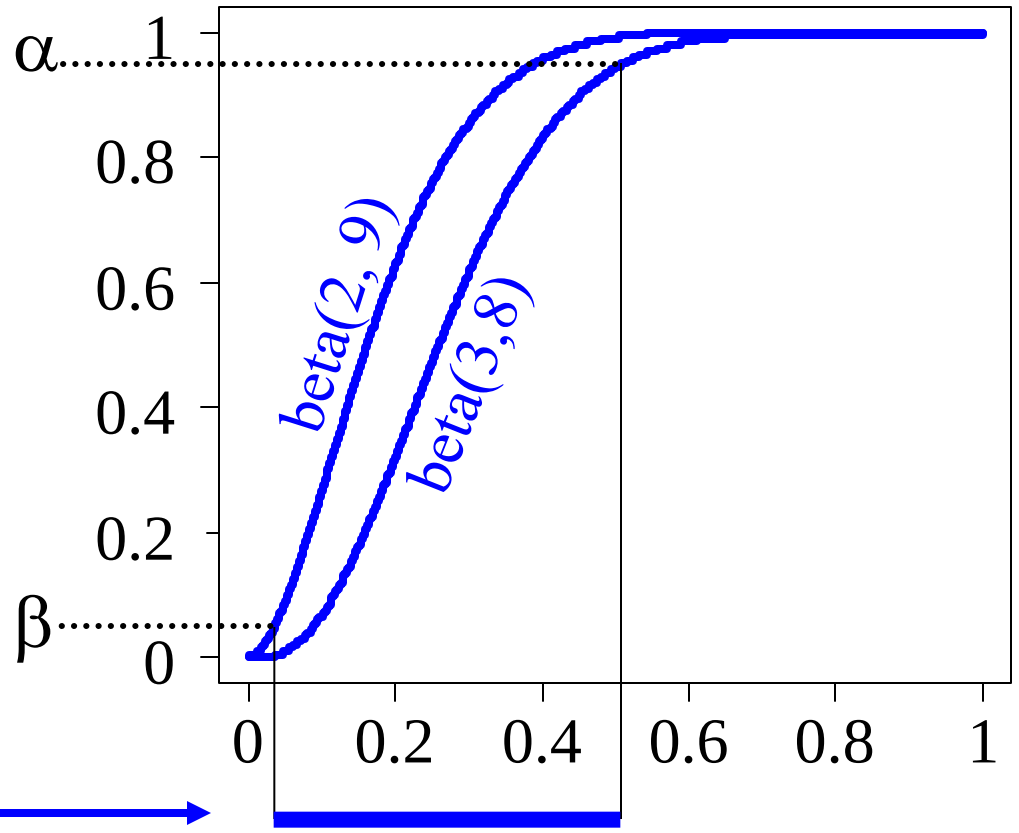
Probability p for k of n trials

$p \sim \text{env}(\text{beta}(k, n-k+1), \text{beta}(k+1, n-k))$

Data

$k = 2$

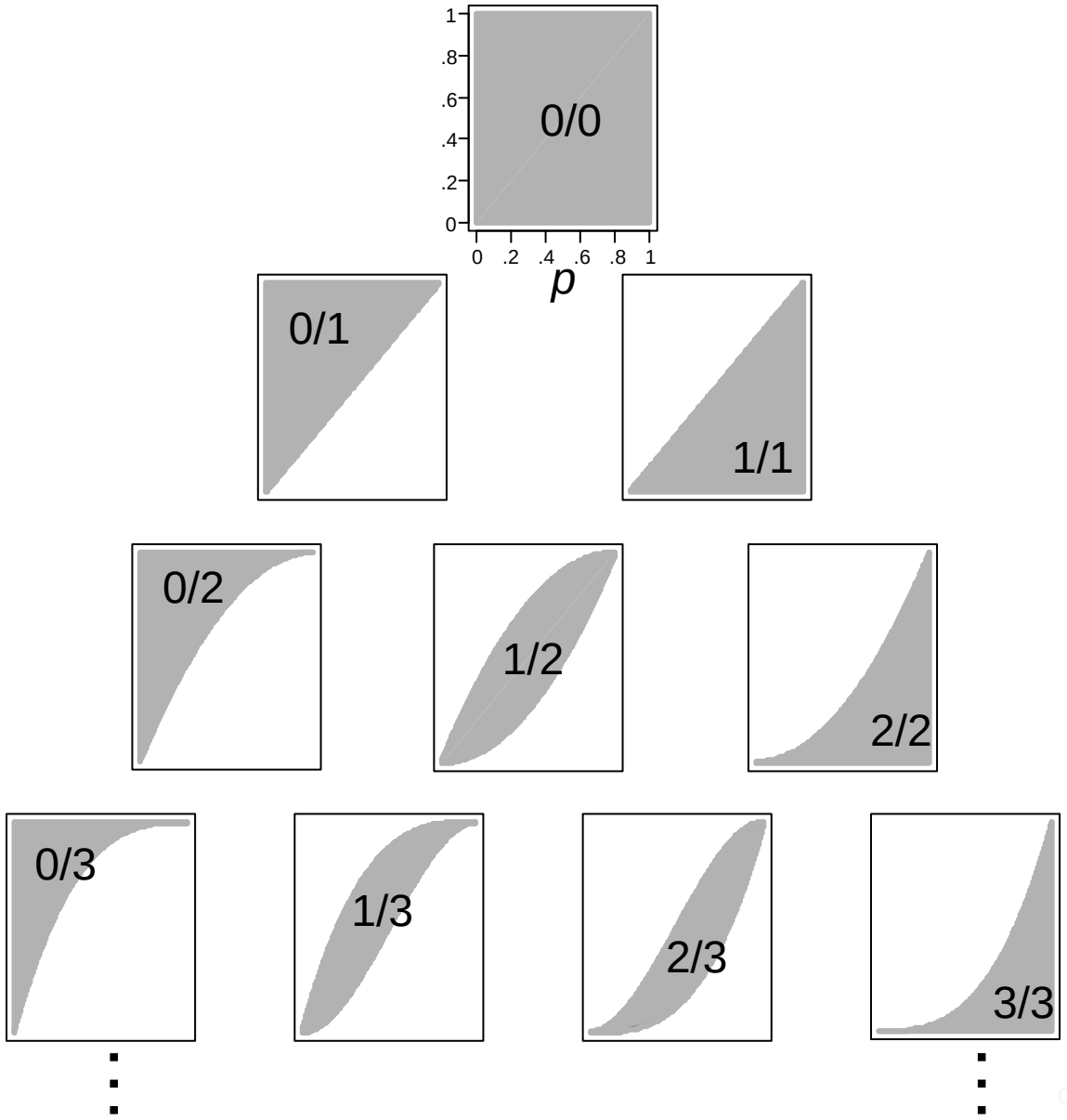
$n = 10$

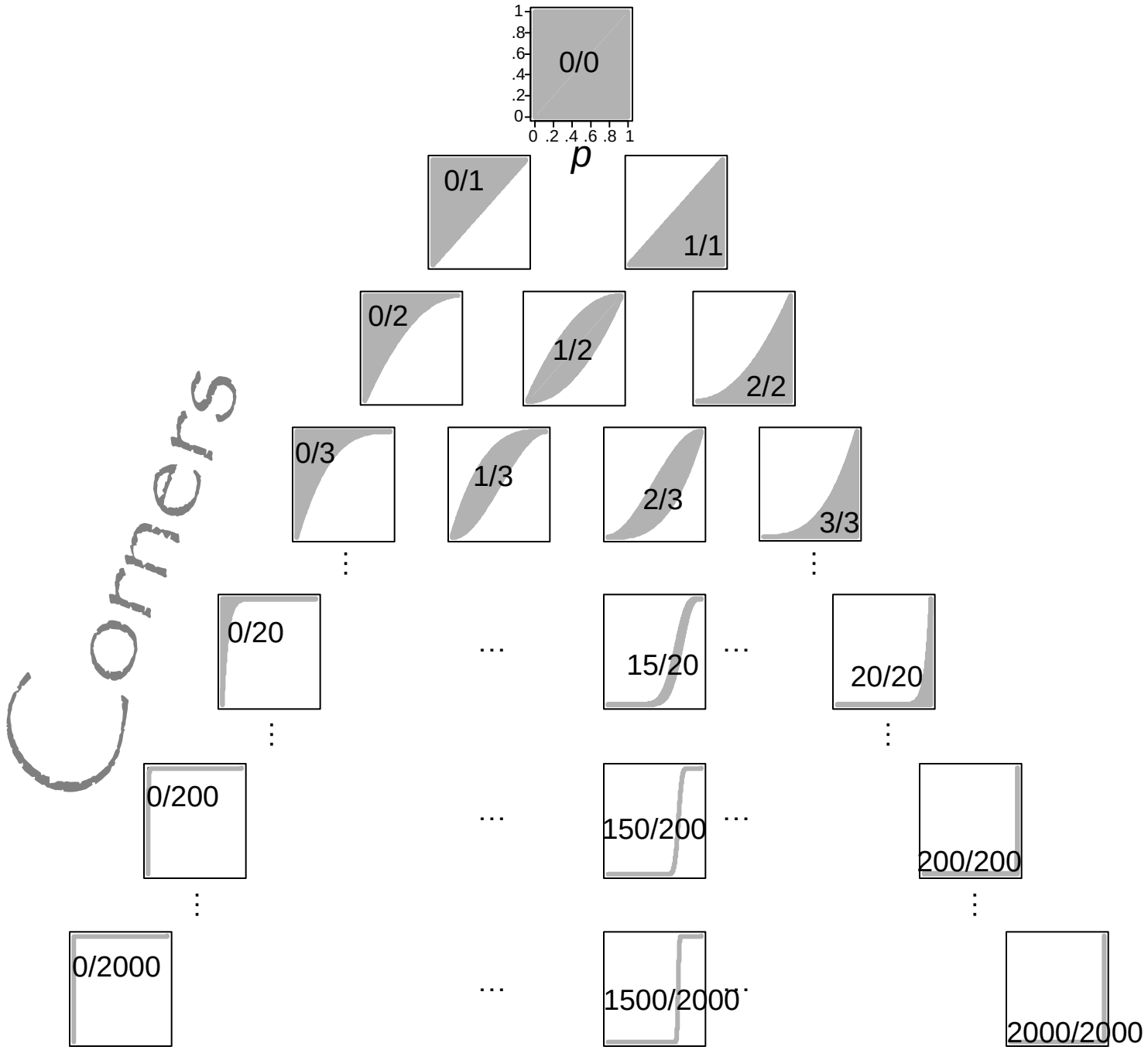


$(\alpha-\beta)100\%$ confidence interval for p →

If $1-\alpha = \beta$, result is identical to classical Clopper–Pearson interval

C-boxes partition the vacuous square

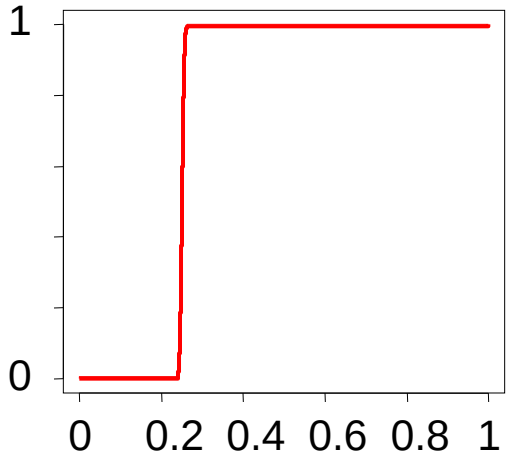



Corners
0/0
0/1
1/1
0/2
1/2
2/2
0/3
1/3
2/3
3/3
0/20
15/20
20/20
0/200
150/200
200/200
0/2000
1500/2000
2000/2000
p

Walley's IBM

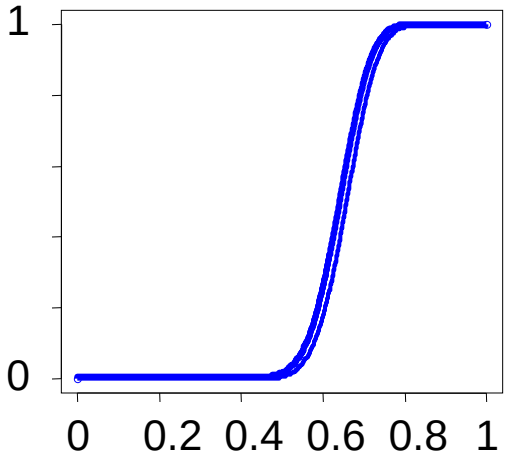
- Assumes beta distributions
  - C-boxes make no shape assumptions
- Needs to select parameter *s*
  - C-boxes have no such parameter
- Works for one problem (binomial *p*)
  - C-boxes are general for many problems
- Not defined by confidence or performance

Computing with c-boxes



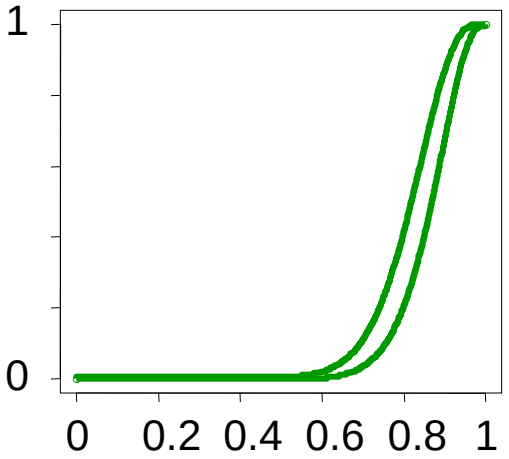
Plan A

249 out of 1,014 failed



Plan B

39 out of 60 failed

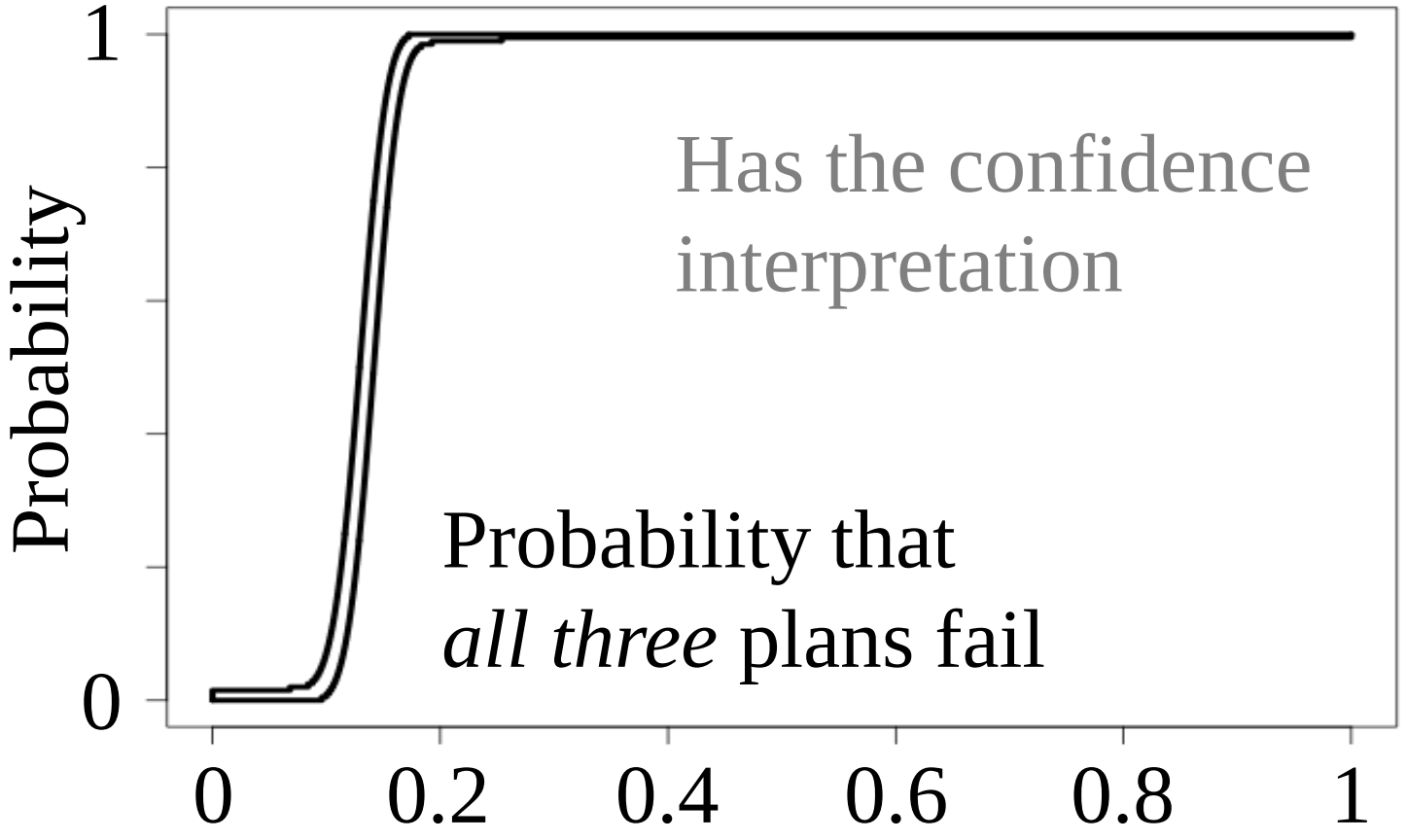


Plan C

17 out of 20 failed

What if we tried all three plans independently?

Conjunction (AND)



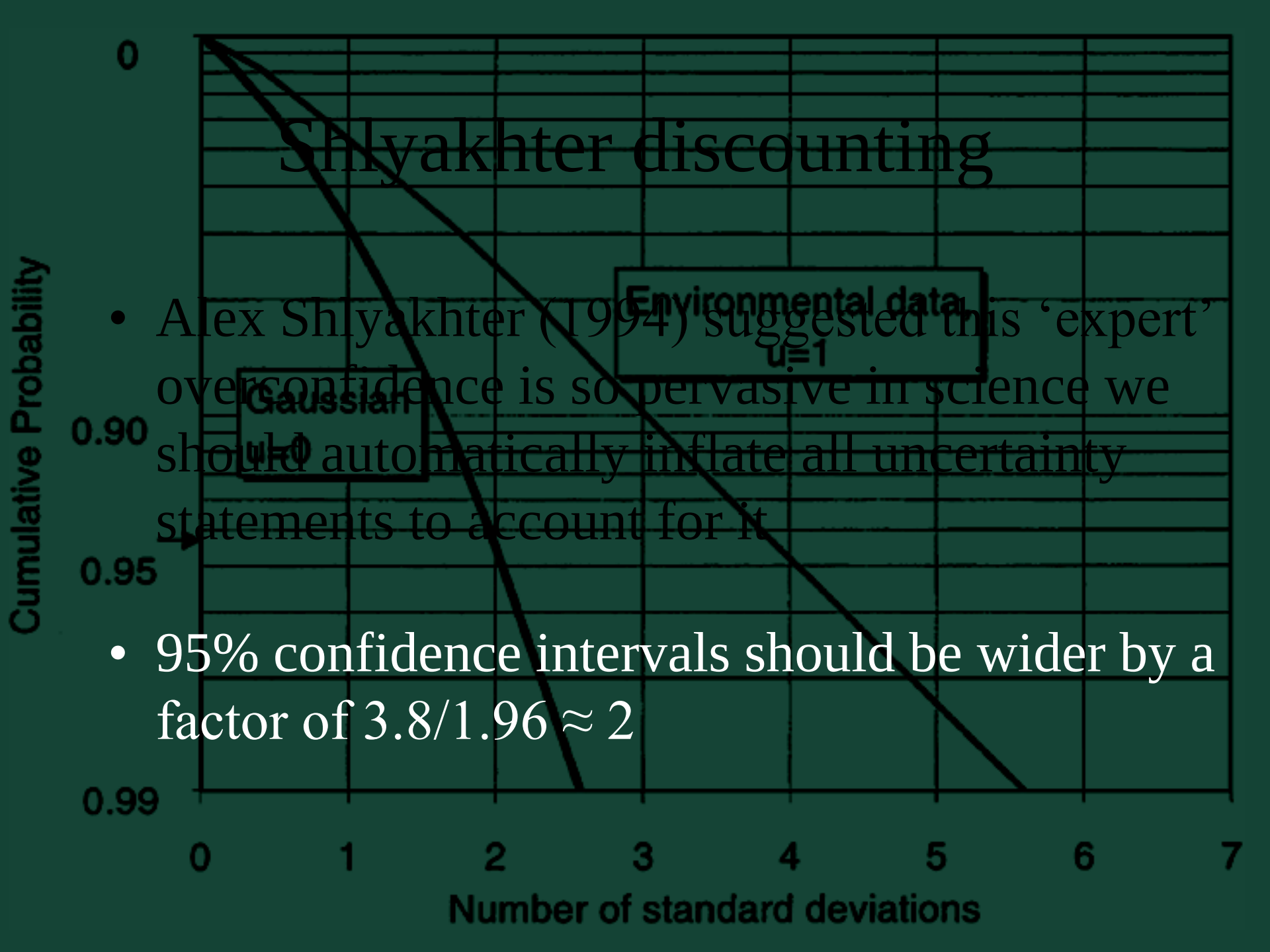
Confidence intervals are clumsy

- Frequentists like confidence intervals but cannot use them in subsequent calculations
- Bayesians can compute with posteriors, but they don’t guarantee statistical performance
- C-boxes take the best from both worlds

Overconfidence

- People, including scientists and engineers, systematically understate their uncertainty
  - 90% confidence intervals ought to enclose the true value 90% of the time on average, but do so only about 30 to 50% of the time
  - Overconfidence “has been found to be almost universal in all measurements of physical quantities”
- Likely to be at least as important in expert elicitation when nothing is being measured

Shlyakhter discounting



- Alex Shlyakhter (1994) suggested this ‘expert’ overconfidence is so pervasive in science we should automatically inflate all uncertainty statements to account for it
- 95% confidence intervals should be wider by a factor of 3.8/1.96 ≈ 2

Elicitation penalty

- Expert opinions aren't like random *data*
- Their uncertainty should be inflated
- The penalty size should be derived empirically from validation studies of prior elicitations

Placeholder for the penalty

- The numbers in “1 in 300” are not counts but estimates with imprecision implied by sigdigs

$$P(\text{“1 in 300”}) = [0.5, 1.5] / [250, 350]$$
$$= [0.5 / 350, 1.5 / 250]$$

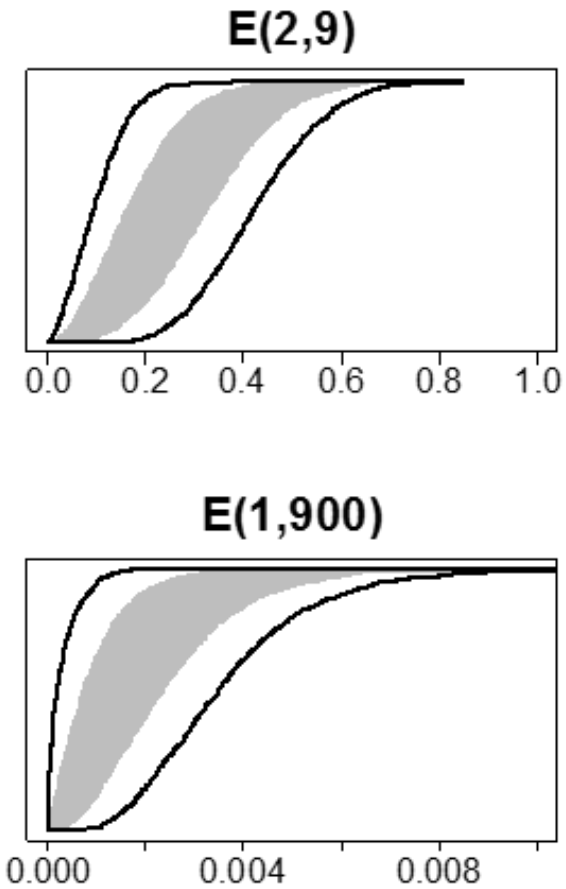
- The *envelope* of all corresponding c-boxes

Significant-digit intervals

- Significant digits imply an interval (± half the magnitude of the last significant decimal place)

x	s(x)
0	[0, 0.5]
1	[0.5, 1.5]
9	[8.5, 9.5]
10	[5, 15]
300	[250, 350]
8150	[8145, 8155]
1×10^{7}	[5×10^{6}, 1.5×10^{7}]

Examples for elicitations "*k* in *n*"



Gray c-boxes B(*k*, *n*), and black envelopes of c-boxes B(s(*k*), s(*n*))

More uncertainty for round numbers

- Doubles (or more than doubles) the uncertainty
- Why *more than* doubles?

  s(9) = [8.5, 9.5] unit width

  s(10) = [5, 15] width of 10

- Presumes greater uncertainty when round numbers are used to characterize a probability

Linguistic uncertainty

- "Words of estimative probability"
- Sherman Kent, Central Intelligence Agency

certain	100%
almost certain	(93 ± ~6)%
probable	(75 ± ~12)%
chances about even	(50 ± ~10)%
probably not	(30 ± ~10)%
almost certainly not	(7 ± ~5)%
impossible	0%

Holes; Not designed for humans (jargon); Never widely used

Hedges

- Words or phrases that modify numbers, often to express uncertainty
  - about...
  - …approximately…
  - …almost…
  - …at least…
  - …at most…
  - …and change
  - …and some
  - around…
- There is knowledge trapped in hedges

Empirical characterisation

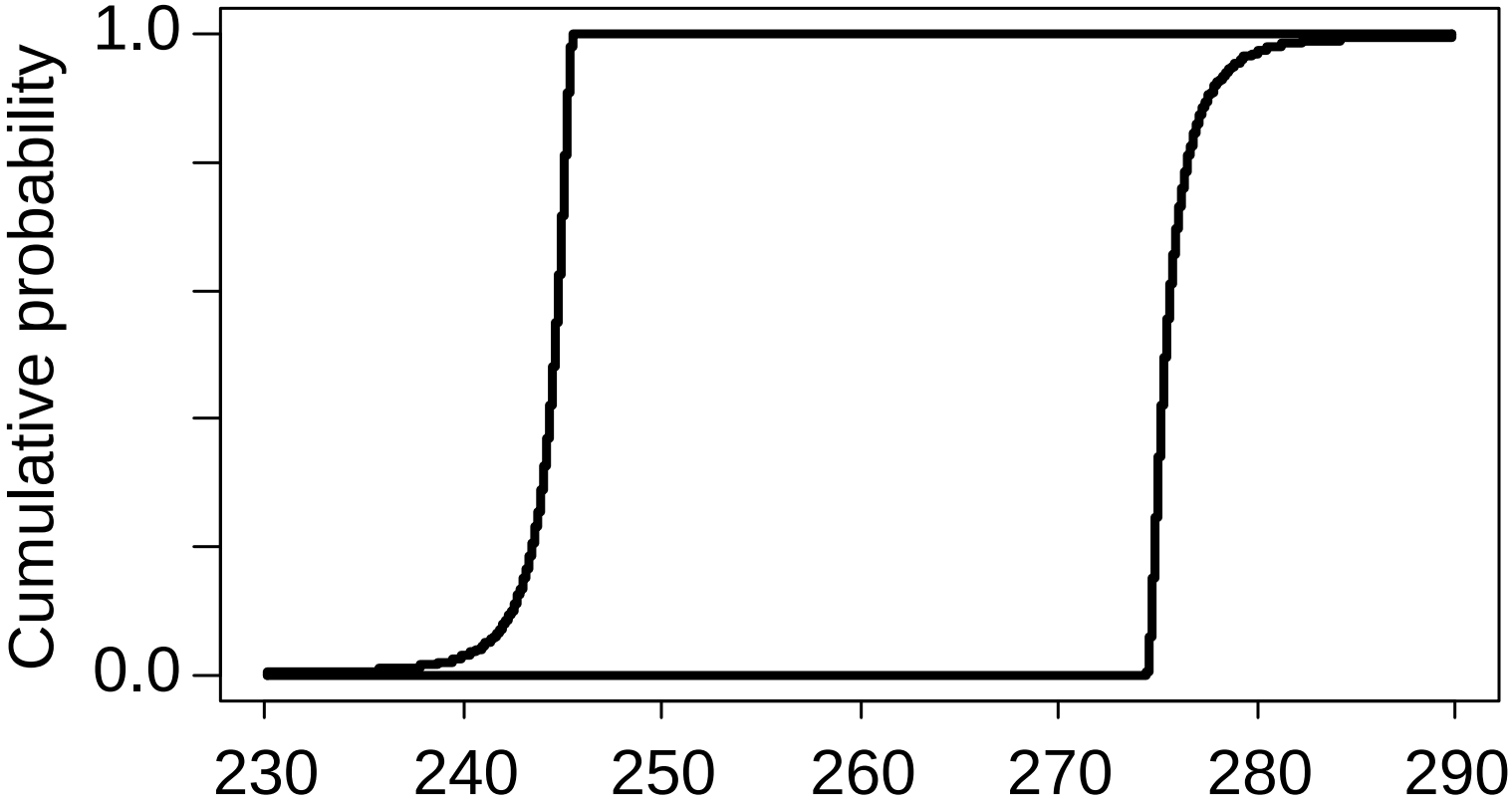
- Questionnaires
  - Makes people dizzy
- Von Ahn “games with a purpose”
  - Encode the question as a game whose solution answers the question
- Amazon Mechanical Turk
  - Pay people to answer questions or do small tasks

Amazon Mechanical Turk

- ~400 turkers, whose native language is English
- < $50
- Multifarious phrasings and contextualisations

- People interpret uncertainty narrowly
- Allow us to *quantitatively* characterise hedges

“About 260”



Future work

- Method will be applied to a real fault tree analysis conducted on a malaria control program involving GMOs
- Combining / pooling expert opinions
- Amazon Mechanical Turk study to derive an empirical uncertainty penalty

Acknowledgments

- Keith Hayes, CSIRO
- Michael Balch, Alexandria Validation
- National Institutes of Health

Confidence distributions

- Not widely used in statistics
- Introduced by Cox in the 1950s
- Closely related to well known ideas
  - Student's *t*-distribution
  - Bootstrap distributions
- Don't exist for the binomial rate

Confidence boxes

- Structures that let you infer confidence intervals for a parameter, at any confidence level
- Can be propagated just like p-boxes
- Allow us to *compute with confidence*

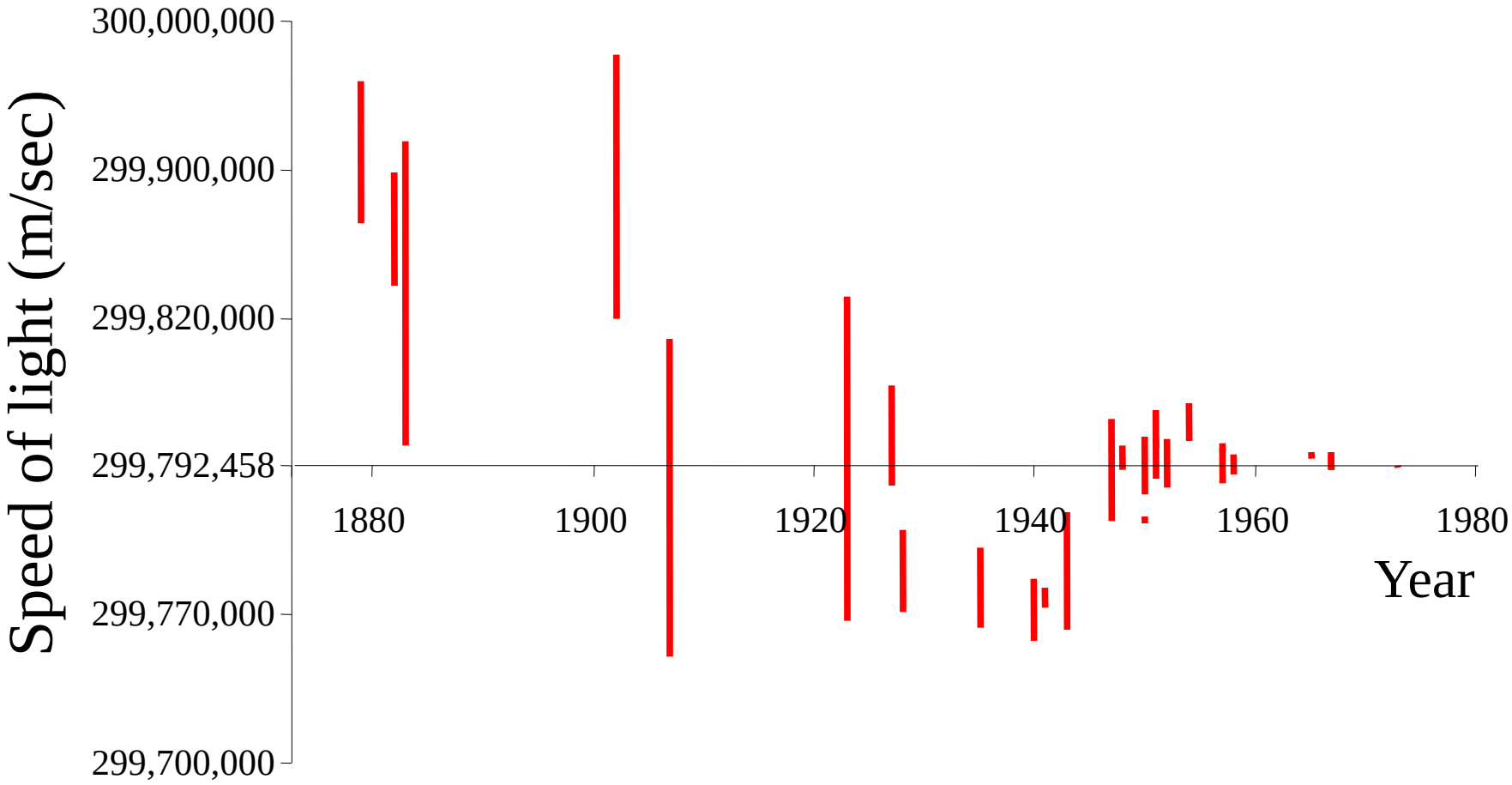
C-boxes

- Not unique
  - Just as confidence intervals are not unique
  - May create some flexibility
- Depend on stopping rule
  - But not knowing the stopping rule may just mean the c-box is wider (and knowing it tightens it)
- Don’t seem overly conservative in practice

Overconfidence

- Humans are too confident
  - Intervals they give are consistently too narrow
  - Stock projections, project timelines, etc., etc.
  - Scientific measurements understate imprecision too
- Empirical evidence documents understatement of uncertainty in measurements of all kinds of physical constants and chemical values
  - Youden, and Morgan and Henrion document many examples

Overconfidence in measuring *c*



http://www.sigma-engineering.co.uk/light/lightindex.shtml